Asian Research Journal of Mathematics

10(2): 1-17, 2018; Article no.ARJOM.43157

ISSN: 2456-477X

On Finite Group Presentations and Function Decomposition Based on Linearity of Discrete-Time Signal

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Authors' contributions

This work was carried out in collaboration between all authors. Author DS designed the study, wrote the GAP Program and wrote the first draft of the manuscript. Authors BAM and SGN authenticate the results and the literature searches. All authors read and approved the final manuscript.

Article Information

DOI: 10.9734/ARJOM/2018/43157
Editor(s):
(1) Wei-Shih Du, Professor, Department of Mathematics, National Kaohsiung Normal University, Taiwan.
Reviewers:
(1) Shpetim Rexhepi, Mother Theresa University, Skopje, Macedonia.
(2) Omar Abu Arqub, Al-Balqa Applied University, Jordan.
(3) Antonio Aparecido de Andrade, University at São José do Rio Preto, Brazil.
Complete Peer review History: http://www.sciencedomain.org/review-history/25838

Original Research Article

Received: 27th May 2018
Accepted: 30th July 2018
Published: 10th August 2018

Abstract

Based on the concept of group representation theory, new representations can be generated by direct product (or tensor product) of any two representations of a group. In such case, their irreducible representations are also the direct product. But the conditions under which these representations can be chosen and how to decompose them is silent. In this work, a clear and efficient method for generating and decomposing representations is presented. The study is restricted to geometric group D_n of order 2n and its subgroups, where a new homomorphism called a transfer function based on the geometric group is constructed. Due to linearity of discrete-time signal, the generated transformations are used on signal space. Thus, a different approach to signal processing with the choice of a group of transformations is established.

Keywords: Finite group; representation; decomposition; Fourier transform; signal processing.

2010 Mathematics Subject Classification: 20C30.

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1 Introduction

The goal of group representation theory is to describe groups via their actions on vector spaces. Consideration of groups acting on sets leads to such important results such as the Sylow theorems. By acting on vector spaces, even more, detailed information about a group can be obtained. This is the subject of representation theory. As byproducts emerged Fourier analysis on finite groups and the study of complex-valued functions on a group.

1.1 Preliminaries

An action of a group G on a set X is the same thing as a homomorphism ϕ: $G \rightarrow S_X$, where S_X is the symmetric group on X [1]. This motivates the following definition.

If K is a field and G is a group, we define a representation of G as the pair (ρ, V) where V is a vector space over K and ρ is a homomorphism of G, given by ρ: $G \rightarrow GL_K(V)$. We also defined a K-algebra as a ring whose underlying Abelian group is a K-vector space such that the multiplication map $R \times R \rightarrow R$ is K-bilinear.

Again let R be a K-algebra. An R-module is a pair (ρ, V) where V is a vector space over K such that $\widetilde{\rho} : R \rightarrow End_K V$. Also a group algebra K[G] is a vector space with basis $\{1_g : g \in G\}$ and multiplication $1_g 1_h = 1_{gh}$.

Equivalently, K[G] is the space of functions f: $G \rightarrow K$ with multiplication as follows:

$$f_1 * f_2(g) = \sum_{xy=g} f_1(x) f_2(y) . \tag{1.1}$$

Definition 1.1.1: Let $r \in R$ and define $\widetilde{L}(r) : R \rightarrow R$ by $x \mapsto r \cdot x$. Then $(\widetilde{L}, R)$ is called the left regular module. If R = K[G], then there exists a representation (L, K[G]) of G, called the left regular representation of $g \in G$ given by $L(g)1_x = 1_{gx}$ [1].

Similarly, right regular representation is given by $R(G)1_x = 1_{xg^{-1}}$.

Invariant Subspace: We say that $W \subset V$ is an invariant subspace of a representation ρ: $G \rightarrow GL_K(V)$ if

$\rho(g)W \subset W$ for all $g \in G$.

For example, a constant function form an invariant subspace of (L, K[G]).

Note that a simple representation (or module) contains no non-trivial proper subspaces. For example, Every 1-dimensional representation is simple.

Definition 1.1.2: Intertwines: Let (ρ_1, V_1) and (ρ_2, V_2) be representations. The linear map $T: V_1 \rightarrow V_2$ is called an intertwiner if

$T(\rho_1(g)v) = \rho_2(g)(T(v))$ or $T \circ \rho_1(g) = \rho_2(g) \circ T$ for all $g \in G$.

Lemma 1.1.3 (Shur's Lemma 1): If K is algebra closed, V is a finite dimensional simple representation of G. Then every self-intertwiner T: $V \rightarrow V$ is a scalar multiple of id_V [1].

Proof: V has a subspace $V_\lambda = \{v \in V : T_v = \lambda v\} \neq 0$ which is an invariant subspace. So $V_\lambda = V$, and therefore, $T = \lambda id_V$.◊

Note: Two spaces V_1 and V_2 are isomorphic if there exists a bijective intertwiner $T: V_1 \rightarrow V_2$ and we write $V_1 \cong V_2$.

Lemma 1.1.4 (Shur's Lemma 2): If V_1 and V_2 are simple, then every non-zero intertwiner is an isomorphism. Consequently, either $V_1 \cong V_2$ or $\mathrm{Hom}_G(V_1, V_2) = 0$ [1].

Corollary 1.1.5: If K is algebra closed, V_1, V_2 simple and $T: V_1 \rightarrow V_2$ a non-zero intertwiner, then $\mathrm{Hom}_G(V_1, V_2) = KT$.

Proof: If $S \in \mathrm{Hom}(V_1, V_2)$, then $T^{-1} \circ S \in \mathrm{End}_K V_1$. Thus, $T^{-1} \circ S = \lambda \mathrm{Id}_{V1}$ and so, $S = \lambda T$.$^{\Diamond}$

We shall write ϕ_g for $\phi(g)$ and $\phi_g(v)$, or $\phi_g v$, for the action of ϕ_g on $v \in V$.

Note: Suppose that dim $V = n$, then for any basis B for V, one can associate a vector space isomorphism T: $V \rightarrow C^n$ by taking coordinates. More precisely, if $B = \{b_1,...,b_n\}$, then $T(b_i) = e_i$, where e_i is the i^{th} standard unit vector. We can then define a representation ψ: $G \rightarrow GL_n(C)$ by setting $\psi_g = T\phi_g T^{-1}$ for $g \in G$. If B' is another basis, we have another isomorphism S: $V \rightarrow C^n$, and hence a representation $\psi' : G \rightarrow GL_n(C)$ is given by $\psi'_g = S\phi_g S^{-1}$. The representations ψ and ψ' are related via the formula $\psi'_g = ST^{-1}\psi_g TS^{-1} = (ST^{-1})\psi_g(ST^{-1})^{-1}$.

2 Review of Relevant Group Theoretical Concepts

In this section, we review the work of Rajathilagam, [2]. A discrete signal is normally viewed as a function $f(t)$, a set of points over time. Any change to the signal may be seen as a change of every point (the domain) in the signal to another point (the range). The pair-wise correspondence of points in the domain and range may be pictured as a mapping [2]:

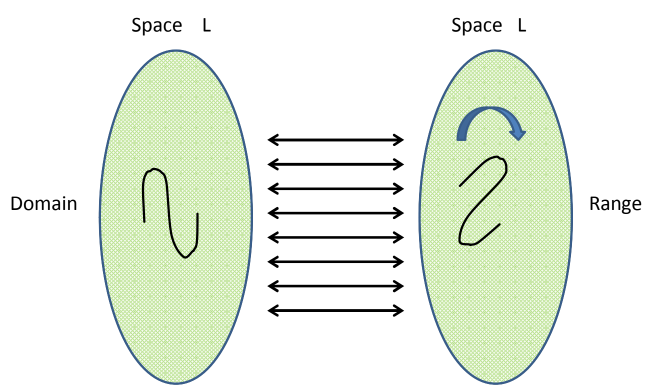

Fig. 1. Mapping points in a vector space L

When the mapping is one-to-one, then it is a transformation, which means every point in the domain corresponds to only one point (image) in the range. The transformation shown in the Fig. 1 is a rotation. Examples of some more transformations include reflection, dilation, scaling and shifting. Transformations do not reduce the size (length) of the signal. Therefore, there is always an inverse transformation to bring back the signal to the original domain. There is also an identity transformation which does not bring any change to the signal by mapping a point to the same point in the domain itself. Further, if two transformations are applied to a signal consecutively, then it is possible to combine the effect of both in a single transformation.

A special case of transformations is linear transformations in n-dimensional space. In a linear transformation, with respect to a fixed coordinate system, a point with coordinates $(v_1, v_2, ..., v_n)$ is mapped to a point with coordinates $(v_1', v_2', ..., v_n')$ where the v_i' coordinates are given by:

$$v_i' = \sum_{i,j=1}^{n} \alpha_{ij} v_i \tag{2.1}$$

The n^2 coefficients α_{ij} form a two dimensional array T_i, which is the matrix of transformation for all points v_i to v_i'. The transformation is invertible if the matrix is invertible. Again an identity and inverse transformation matrix may be defined in the n-dimensional space for points v_i.

A group of transformations may be defined as a set of transformations on a given point set or any set of elements which:

i) a law of composition is given so that the composition or product of two transformations T_1T_2 is well defined. If T_1 and T_2 belong to the group, then T_1T_2 also belongs to the group.
ii) all the transformations obey the associative law, ie., $T_1(T_2T_3) = (T_1T_2)T_3$.
iii) has an identity transformation T_e satisfying $T_eT_n = T_nT_e = T_{ne=} T_n$.
iv) for every transformation T, there is an inverse transformation T^{-1} such that

$$TT^{-1} = T_e = T^1T \text{ [3].}$$

Some examples of transformation groups are:

i) The set of all n! permutations of a set of n points.
ii) The transformations of an equilateral triangle which bring the triangle to coincide with itself.
iii) Rotations of three dimensional space through angles 0°, 120° and 240° about the z-axis.

Two transformations of a group are said to be 'similar' if they change only the position and size of an object they transform but not its shape. A dilation-rotation, reflection-rotation, etc., are examples of similarity transformations pair. In group theory, similarity transformations are called conjugate transformations. Two elements of a group G, T_1 and T_2 are called conjugate elements if another element $T \in G$ exists such that

$$TT_1T^{-1} = T_2. \tag{2.2}$$

For a better understanding, the geometrical meaning of the conjugate transformations may be looked at. Let T_1 and T_2 be rotations of a group G. Let T be taken as a reflection. If T_1 is a rotation around an axis X, then T_2 is the same rotation around another axis X' obtained by reflecting the axis X by T. This may be written as

$$X' = TX. \tag{2.3}$$

2.1 Transformations

Fourier series generate a representation for a signal in terms of sine and cosine functions, giving an infinite set of basis vectors for the signal. Here time and frequency information cannot be localized on specific parts of the signal. Generalized Fourier basis is formed by creating representation matrices of the transformations and calculating their trace, which is called the characters of the transform. Wavelets [4,5] create orthogonal subspaces using the transformations dilation and translation to capture different frequencies of the signal. Still, due to the edge effect produced by breaking frequencies into smaller and smaller pieces, in applications like face recognition they are unable to testify a good match.

Ridgelet transform [6,7] focuses on detecting lines in an image using the Radon transform [8]. Curves are specially recognized in a signal by curvelets [9], which is a combination of wavelets and ridgelets. These two transforms are specialized forms of wavelets. There are also other specialized forms of wavelets like the contourlets [10], wedgelets [11], and grouplets [12]. In grouplets, the wavelet coefficients are grouped together based on the direction of their flow using multiscale association fields. Representation theory using rotations and reflections have been used by Lenz [13,14] for edge detection in images but they do not generalize the use of transformation groups to generate an orthogonal basis. In a recent work by Lenz, octahedral transformation group has been chosen to be suitable for a specific application, i.e., the 3-D image environment, and representations have been used to describe this environment by evaluating their correlation coefficients. Vale, 2008 [15] shows how an orthonormal set of polynomials can be generated by the orbit of a vector using representation theory [16,17]. In this work, we focus on decomposing transforms of a transformation group such as rotations and reflections through their irreducible representations. Dihedral groups have been chosen for the illustrations.

2.2 Fourier transform

In this section, we review and introduce an algebraic structure on an operator $L(G)$ coming from the convolution product. The Fourier transform can be used to analyze this structure more clearly in terms of known rings.

2.2.1 Periodic functions

We begin with the classical case of periodic functions on the integers.

Definition 2.2.1 (Periodic function)**.** A function $f : Z \to C$ is said to be *periodic* with period n if $f(x) = f(x + n)$ for all $x \in Z$.

Notice that if n is a period for f, then so is any multiple of n. It is easy to see that periodic functions with period n are in bijection with elements of $L(Zn)$, that is, functions f: $Z_n \to C$. Indeed, the definition of a periodic function says precisely that f is constant on residue classes modulo n. Now, the irreducible characters form a basis for $L(Z_n)$. It follows that if f: $Z_n \to C$ is a function, then

$$f = \langle \chi_0, f \rangle \chi_0 + \cdots + \langle \chi_{n-1}, f \rangle \chi_{n-1}. \tag{2.4}$$

The Fourier transform encodes this information as a function.

For example, consider a function

$$f(n) = \cos(6\pi / 5n).$$

With $n \in [-10:10]$, the function is periodic with period 5. We therefore construct a digital signal for $f(n)$ as follows.

```
>> n = [-10:10];
>> x = cos(6*pi/5*n);
>> stem(n,x)
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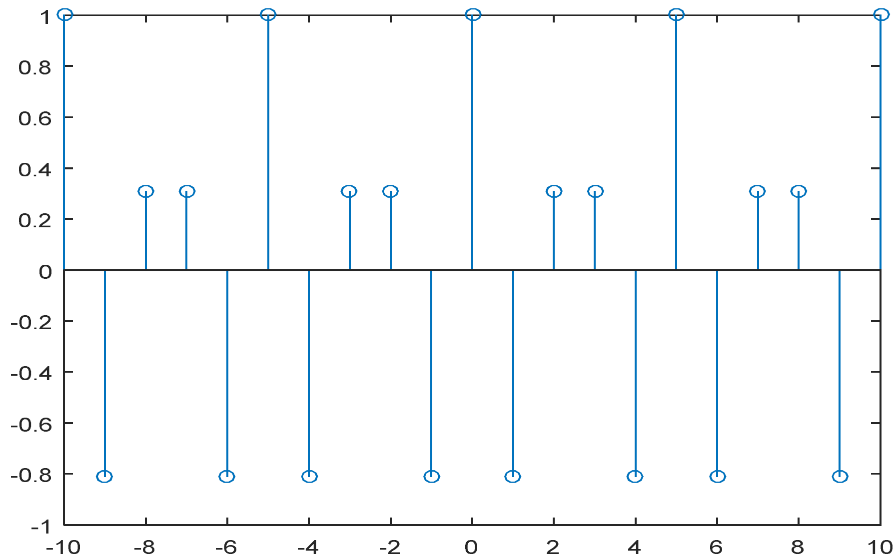

Fig. 4. Periodicity in signal

Definition 2.2.2 (Fourier transform)**.** Let f: $Z_n \to C$. Then the Fourier transform $\hat{f}: Z_n \to C$ of f, is defined by

$$\hat{f}(\overline{m}) = n\langle \chi_m, f \rangle = \sum_{k=0}^{n-1} e^{-2\pi imk} f(\overline{k}). \tag{2.5}$$

It is immediate that the Fourier transform is a linear transformation T: $L(Z_n) \to L(Z_n)$ by the linearity of inner products in the second variable. We can rewrite the equation 2.2 as:

Proposition 2.2.3 (Fourier inversion). The Fourier transform is invertible. More precisely,

$$f = \frac{1}{n}\sum_{k=0}^{n-1} \hat{f}(\overline{k})\chi_k . \tag{2.6}$$

The Fourier transform on cyclic groups is used in signal and image processing. The idea is that the values $\hat{f}$ to the wavelengths associated to the wave function f, which is used in compressing the wave. To recover something close enough to the original wave, as far as our eyes and ears are concerned, one applies Fourier inversion.

2.2.2 The convolution product

We now introduce the convolution product on $L(G)$, thereby explaining the terminology group algebra for $L(G)$.

Definition 2.2.4 (Convolution)**.** Let G be a finite group and $\alpha, \beta \in L(G)$. Then the convolution $\alpha * \beta$: $G \to C$ is defined by

$$\alpha * \beta(x) = \sum_{y \in G} \alpha(xy^{-1})\beta(y). \tag{2.7}$$

The convolution gives $L(G)$ the structure of a ring and each element $g \in G$ is associated with the delta function δ_g. Also by the definition, a multiplication $*$ on $L(G)$ satisfy $\delta_g * \delta_h = \delta_{gh}$. To see this, indeed,

$$\delta_g * \delta_h(x) = \sum_{y \in G} \delta_g(xy^{-1})\delta_h(y), \tag{2.8}$$

and the only non-zero term is when $y = h$ and $g = xy^{-1} = xh^{-1}$, i.e., $x = gh$. In this case, we get 1, so we have proved:

Proposition 2.2.5 For $g,h \in G$, $\delta_g * \delta_h = \delta_{gh}$.

Now if $\alpha,\beta \in L(G)$, then $\alpha = \sum_{y \in G} \alpha(g)\delta_g$, $\beta = \sum_{y \in G} \beta(g)\delta_g$ and so if $L(G)$ is a ring, then the distributive law would yield

$$\alpha * \beta = \sum_{g,h \in G} \alpha(g)\beta(h)\delta_g * \delta_h = \sum_{g,h \in G} \alpha(g)\beta(h)\delta_{gh} . \tag{2.9}$$

Applying the change of variables $x = gh$, $y = h$ then

$$\alpha * \beta = \sum_{x \in G} \left(\sum_{y \in G} \alpha(xy^{-1})\beta(y) \right) \delta_x \tag{2.10}$$

which is equivalent to the formula (2.7).

Theorem 2.2.6 The set $L(G)$ is a ring with addition taken pointwise and convolution as multiplication. Moreover, δ_1 is a multiplicative identity.

Proof. It is sufficient to prove δ_1 is the identity and the associativity of convolution holds in $L(G)$. Let $\alpha \in L(G)$. Then

$$\alpha * \delta_1(x) = \sum_{y \in G} \alpha(xy^{-1})\delta_1(y^{-1}) = \alpha(x), \tag{2.11}$$

since $\delta_1(y^{-1}) = 0$ except when $y = 1$. Similarly, $\delta_1 * \alpha = \alpha$. This proves δ_1 is the identity.

For associativity, let $\alpha,\beta,\gamma \in L(G)$. Then

$$[(\alpha * \beta) * \gamma](x) = \sum_{y \in G} [\alpha * \beta](xy^{-1})\gamma(y) = \sum_{y \in G} \sum_{z \in G} \alpha(xy^{-1}z^{-1})\beta(z)\gamma(y) \tag{2.12}$$

We make the change of variables $u = zy$ (and so $y^{-1}z^{-1} = u^{-1}$, $z = uy^{-1}$). The right hand side of the equation 2.12 then becomes

$$\sum_{y \in G} \sum_{u \in G} \alpha(xu^{-1})\beta(uy^{-1})\gamma(y) = \sum_{u \in G} \alpha(xu^{-1}) \sum_{y \in G} \beta(uy^{-1})\gamma(y)$$

$$= \sum_{u \in G} \alpha(xu^{-1})[\beta * \gamma](u)$$

$$= [\alpha * (\beta * \gamma)](x),$$

completing the proof.$^{\Diamond}$

It is now high time to justify the notation $Z(L(G))$ for the space of class functions on G. Recall that the center $Z(R)$ of a ring R consists of all elements $a \in R$ such that $ab = ba$ for all $b \in R$. For instance, the scalar matrices form the center of $M_n(C)$.

Proposition 2.2.7 $Z(L(G))$ is the center of $L(G)$. That is, $f: G \to C$ is a class function if and only if $\alpha * f = f * \alpha$ for all $\alpha \in L(G)$.

Proof. Suppose first that f is a class function and let $\alpha \in L(G)$. Then

$$\alpha * f(x) = \sum_{y \in G} \alpha(xy^{-1}) f(y) = \sum_{y \in G} \alpha(xy^{-1}) f(xyx^{-1}), \tag{2.13}$$

since f is a class function. Setting $z = xy^{-1}$ turns the right hand side of (2.5) into

$$\sum_{z \in G} \alpha(z) f(xz^{-1}) = \sum_{z \in G} f(xz^{-1}) \alpha(z) = f * \alpha(x).$$

And hence, $\alpha * f = f * \alpha$

For the other direction, let f be in the center of $L(G)$.

Claim. $f(gh) = f(hg)$ for all $g,h \in G$.

Proof of claim. Observe that

$$f(gh) = \sum_{y \in G} f(gy^{-1}) \delta_{h^{-1}}(y) = f * \delta_{h^{-1}}(g)$$

$$\delta_{h^{-1}} * f(g) = \sum_{y \in G} \delta_{h^{-1}}(gy^{-1}) f(y) = f(hg),$$

since $\delta_{h^{-1}}(gy^{-1})$ is non-zero if and only if $gy^{-1} = h^{-1}$, that is, $y = hg$.

To complete the proof, we note that by the claim, $f(ghg^{-1}) = f(gg^{-1}h) = f(h)$, establishing that f is a class function. ◊

Example 2.2.8 (Periodic functions on Z)**.** Let f,g: $Z \to C$ be periodic functions with period n. Then convolution is defined by

$$f * g(m) = \sum_{k=0}^{n-1} f(m-k) g(k).$$

The Fourier transform is then

$$\bar{f}(m) = \sum_{k=0}^{n-1} e^{-2\pi imk/n} f(k).$$

Again, by Fourier inversion theorem, we have

$$f(m)=\frac{1}{n}\sum_{k=0}^{n-1}e^{2\pi imk/n}\bar{f}(k).$$

By multiplication formula, $\overline{f*g}=\bar{f}\cdot\overline{g}$

The next proposition shows that T is a vector space isomorphism.

Proposition 2.2.9 The map T: $L(G)\to M_{d1}(C)\times\cdots\times M_{ds}(C)$ is a vector space isomorphism.

Proof. To show that T is linear, it suffices to prove

$$\left(c_1\widehat{f_1+c_2}f_2\right)\left(\varphi^{(k)}\right)=c_1\widehat{f_1}(\varphi^{(k)}+c_2\widehat{f_2}(\varphi^{(k)})$$

for $1\le k\le s$. Indeed,

$$\left(c_1\widehat{f_1+c_2}f_2\right)\left(\varphi^{(k)}\right)=\sum_{g\in G}\overline{\varphi_g^{(k)}}\,(c_1f_1+c_2f_2)(g)$$

$$=c_1d\sum_{g\in G}\overline{\varphi_g^{(k)}}f_1(g)+c_2\sum_{g\in G}\overline{\varphi_g^{(k)}}f_2(g)$$

$$=c_1\widehat{f_1}(\varphi_g^{(k)})+c_2\widehat{f_2}(\varphi_g^{(k)}),$$

as was to be proved. The Fourier inversion theorem implies that T is injective. Since

$$\dim L(G)=|G|=d_1^2+\ldots+d_s^2=\dim M_{d_1}(C)\times\ldots\times M_{d_s}(C),$$

it follows that T is an isomorphism. $^{\Diamond}$

This leads us to a special case of a more general theorem of Wedderburn that is often used as the starting point for studying the representation theory of finite groups.

Theorem 2.2.10 (Wedderburn)**.** The Fourier transform

$$T: L(G)\to M_{d1}(C)\times\cdots\times M_{ds}(C)$$

is an isomorphism of rings.$^{\Diamond}$

Now, it can be deduced from Proposition 2.2.6 that T is an isomorphism of vector spaces. Also, if $T(\alpha*\beta)=T\alpha\cdot T\beta$, then T is a ring isomorphism.

Moreover, by the definition of multiplication in direct product,

$$\widehat{\alpha*\beta}\left(\varphi^{(k)}\right)=\widehat{\alpha}(\varphi^{(k)})\cdot\widehat{\beta}(\varphi^{(k)})\text{ for }1\le k\le s. \tag{2.14}$$

For non-abelian groups, it is still true that computing $T\alpha\cdot T\beta$ and inverting T can sometimes be faster than computing $\alpha*\beta$ directly.

3 Main Result

3.1 Transformation decomposition

In this section, a new homomorphism called a transfer function is constructed. This is a mathematical function that gives all possible output values for every corresponding input.

But we shall first introduce the idea of "Transversal" of a subgroup in a group using group of transformations. Throughout this section, the group $G = D_n$, a symmetry group of order $2n$.

Now, let G be a group and H be a non-trivial subgroup of G such that $[G: H] = r$, where $r \in Z^+$ and $r \leq n$. Then a collection T which contain exactly one representative from each coset (left or right) of H in G is called the transversal of H in G. Note that if $T = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ and T is a transversal of H in G, then

$$G = \bigcup_{i=1}^{n} \alpha_i H .$$

Also, $\alpha_i H \bigcap \alpha_j H = \Phi$ for $i \neq j$ and whenever $\beta \in \alpha H$, then $\beta H = \alpha H$. Again, given any finite set X and a bijection f:$X \rightarrow X$, a collection of all such bijections on X is called a symmetric group on X denoted by S_n of order n! Thus if $\sigma \in S_n$, then σ is considered as a transformation on S_n.

We shall now establish some facts about transversals of a subgroup in a group.

Lemma 3.1: Let H be a non-trivial Abelian subgroup of a group G such that $[G: H] = r$. If $T = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ is a transversal of H in G and $g \in G$, then

i) there exists $h_r \in H$ and $\delta \in S_n$ depending on g such that $g\alpha_r = \alpha_{r\delta} h_r$;
ii) if $\{\beta_1, \beta_2, ..., \beta_n\}$ is another transversal of H in G and $k_r \in H$, $\delta \in S_n$, then

$$\prod_{r=1}^{n} h_r = \prod_{r=1}^{n} k_r .$$

Proof: (i) Since $\alpha_r \in H$, $g\alpha_r \in xH$ for some x. Thus, we can find $h_r \in H$ and for a positive integer r, there exists q_r such that $g\alpha_r = \alpha_{q_r} h_r$ where $r \leq n$. Now, to show that q_r defines a transformation in S_n, it is sufficient to show that q_r is injective.

Suppose $g\alpha_{r'} = \alpha_{q_{r'}} h_{r'}$ for $h_{r'} \in H$ and $q_r = q_{r'}$, then we have

$$\alpha_r^{-1}\alpha_{r'} = (g\alpha_r)^{-1}(g\alpha_{r'}) = (\alpha_{q_r} h_r)^{-1}(\alpha_{q_{r'}} h_{r'}) = h_r^{-1} h_{r'} \in H .$$

This implies that

$$\alpha_r H = \alpha_{r'} H \text{ (since } h_r^{-1} \in H,\ h_{r'}^{-1} \in H \text{)}.$$

Hence, $r = r'$ since T is a transversal. This shows that the transformation q is injective and thus, a bijection (since it is an injection of a finite set to itself) of n-elements. Thus, $q \in S_n$.

(ii) Suppose now that $q = \delta \in S_n$. Then from (i) with $g = e$, there exists $p_r \in H$ and $\xi \in S_n$ such that $\beta_r = \alpha_{r\xi} p_r$ for $1 \le r \le n$ and using the identity $\alpha_{r\xi\delta} = \alpha_{r\xi\delta\xi\xi^{-1}}$, it follows that

$$g\beta_r = g(\alpha_{r\xi} p_r) = (\alpha_{r\xi\delta} h_{r\xi}) p_r = \beta_{r\xi\delta\xi^{-1}} (p_{r\xi\delta\xi^{-1}})^{-1} h_{r\xi} p_r .$$

Then since $\xi\delta\xi^{-1} \in S_n$ and $(p_{r\xi\delta\xi^{-1}})^{-1} h_{r\xi} p_r \in H$, set $k = \xi\delta\xi^{-1}$ and $f_r = (p_{r\xi\delta\xi^{-1}})^{-1} h_{r\xi} p_r$. Thus, $g\beta_r = \beta_{rk} f_r$ as required and since H is Abelian,

$$\prod\nolimits_r f_r = \prod\nolimits_r (p_{r\xi\delta\xi^{-1}})^{-1} h_{r\xi} p_r = \prod\nolimits_r p_r^{-1} \prod\nolimits_r h_r \prod\nolimits_r p_r = \prod\nolimits_r h_r$$

since $\xi, \delta \in S_n$ where $r = 1, 2, \ldots, n$. Hence, the result follows.◊

Now, from the construction above, we defined a transfer as a homomorphism from G to H as follows.

Definition 3.2: Let G be a finite group (not necessarily Abelian) and let H be an Abelian subgroup of G such that $[G: H] = r$. With the notation set up in Lemma 5.1, the function $\xi : G \to H$ defined by $g\xi = \prod_{r=1}^{n} h_r$ for all $g \in G$ is called the transfer of H in G.

This is independent of the coset representatives h_r and therefore, the function is well-defined. Next, we shall show that the transfer ξ is a homomorphism.

Theorem 3.3: Let G be a finite group and H be a subgroup of G such that $[G: H] = r$. Then the transfer of H in G, given by $\xi : G \to H$ is a homomorphism.

Proof: Let $g, q \in G$, $g\alpha_r = \alpha_{r\delta} p_r$ and $q\alpha_r = \alpha_{r\sigma} k_r$ and $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a transversal. Again, with $\delta, \sigma \in S_n$,

$$gq\alpha_r = g\alpha_{r\sigma} k_r = \alpha_{r\sigma\delta} p_{r\sigma} k_r .$$

Thus, since H is Abelian,

$$gq\xi = \prod\nolimits_r p_{r\sigma} k_r = \prod\nolimits_r p_r \cdot \prod\nolimits_r k_r = g\xi \cdot q\xi .$$

Hence, ξ is a homomorphism.◊

Example 3.4: Let $G = D_{10} = \left\langle \alpha, \beta : \alpha^{10} = \beta^2 = i, \beta\alpha\beta^{-1} = \alpha^{-1} \right\rangle$. Then we construct the group G as follows:

$$G = \{i, \alpha, \alpha^2, \alpha^3, \alpha^4, \alpha^5, \alpha^6, \alpha^7, \alpha^8, \alpha^9, \beta, \alpha\beta, \alpha^2\beta, \alpha^3\beta, \alpha^4\beta, \alpha^5\beta, \alpha^6\beta, \alpha^7\beta, \alpha^8\beta, \alpha^9\beta\}$$

And $G \times G = \prod_{r=1}^{20} x_r$ where $x \in G$ the operation is the product of permutation. Hence, the following group table for $G \times G$.

Table 1. Group table of D_{10}

$\times$	i	α	α^2	α^3	α^4	α^5	α^6	α^7	α^8	α^9	β	$\alpha\beta$	$\alpha^2\beta$	$\alpha^3\beta$	$\alpha^4\beta$	$\alpha^5\beta$	$\alpha^6\beta$	$\alpha^7\beta$	$\alpha^8\beta$	$\alpha^9\beta$
i	i	α	α^2	α^3	α^4	α^5	α^6	α^7	α^8	α^9	β	$\alpha\beta$	$\alpha^2\beta$	$\alpha^3\beta$	$\alpha^4\beta$	$\alpha^5\beta$	$\alpha^6\beta$	$\alpha^7\beta$	$\alpha^8\beta$	$\alpha^9\beta$
α	α	α^2	α^3	α^4	α^5	α^6	α^7	α^8	α^9	i	$\alpha\beta$	$\alpha^2\beta$	$\alpha^3\beta$	$\alpha^4\beta$	$\alpha^5\beta$	$\alpha^6\beta$	$\alpha^7\beta$	$\alpha^8\beta$	$\alpha^9\beta$	β
α^2	α^2	α^3	α^4	α^5	α^6	α^7	α^8	α^9	i	α	$\alpha^2\beta$	$\alpha^3\beta$	$\alpha^4\beta$	$\alpha^5\beta$	$\alpha^6\beta$	$\alpha^7\beta$	$\alpha^8\beta$	$\alpha^9\beta$	β	$\alpha\beta$
α^3	α^3	α^4	α^5	α^6	α^7	α^8	α^9	i	α	α^2	$\alpha^3\beta$	$\alpha^4\beta$	$\alpha^5\beta$	$\alpha^6\beta$	$\alpha^7\beta$	$\alpha^8\beta$	$\alpha^9\beta$	β	$\alpha\beta$	$\alpha^2\beta$
α^4	α^4	α^5	α^6	α^7	α^8	α^9	i	α	α^2	α^3	$\alpha^4\beta$	$\alpha^5\beta$	$\alpha^6\beta$	$\alpha^7\beta$	$\alpha^8\beta$	$\alpha^9\beta$	β	$\alpha\beta$	$\alpha^2\beta$	$\alpha^3\beta$
α^5	α^5	α^6	α^7	α^8	α^9	i	α	α^2	α^3	α^4	$\alpha^5\beta$	$\alpha^6\beta$	$\alpha^7\beta$	$\alpha^8\beta$	$\alpha^9\beta$	β	$\alpha\beta$	$\alpha^2\beta$	$\alpha^3\beta$	$\alpha^4\beta$
α^6	α^6	α^7	α^8	α^9	i	α	α^2	α^3	α^4	α^5	$\alpha^6\beta$	$\alpha^7\beta$	$\alpha^8\beta$	$\alpha^9\beta$	β	$\alpha\beta$	$\alpha^2\beta$	$\alpha^3\beta$	$\alpha^4\beta$	$\alpha^5\beta$
α^7	α^7	α^8	α^9	i	α	α^2	α^3	α^4	α^5	α^6	$\alpha^7\beta$	$\alpha^8\beta$	$\alpha^9\beta$	β	$\alpha\beta$	$\alpha^2\beta$	$\alpha^3\beta$	$\alpha^4\beta$	$\alpha^5\beta$	$\alpha^6\beta$
α^8	α^8	α^9	i	α	α^2	α^3	α^4	α^5	α^6	α^7	$\alpha^8\beta$	$\alpha^9\beta$	β	$\alpha\beta$	$\alpha^2\beta$	$\alpha^3\beta$	$\alpha^4\beta$	$\alpha^5\beta$	$\alpha^6\beta$	$\alpha^7\beta$
α^9	α^9	i	α	α^2	α^3	α^4	α^5	α^6	α^7	α^8	$\alpha^9\beta$	β	$\alpha\beta$	$\alpha^2\beta$	$\alpha^3\beta$	$\alpha^4\beta$	$\alpha^5\beta$	$\alpha^6\beta$	$\alpha^7\beta$	$\alpha^8\beta$
β	β	$\alpha^9\beta$	$\alpha^8\beta$	$\alpha^7\beta$	$\alpha^6\beta$	$\alpha^5\beta$	$\alpha^4\beta$	$\alpha^3\beta$	$\alpha^2\beta$	$\alpha\beta$	i	α^9	α^8	α^7	α^6	α^5	α^4	α^3	α^2	α
$\alpha\beta$	$\alpha\beta$	β	$\alpha^9\beta$	$\alpha^8\beta$	$\alpha^7\beta$	$\alpha^6\beta$	$\alpha^5\beta$	$\alpha^4\beta$	$\alpha^3\beta$	$\alpha^2\beta$	α	i	α^9	α^8	α^7	α^6	α^5	α^4	α^3	α^2
$\alpha^2\beta$	$\alpha^2\beta$	$\alpha\beta$	β	$\alpha^9\beta$	$\alpha^8\beta$	$\alpha^7\beta$	$\alpha^6\beta$	$\alpha^5\beta$	$\alpha^4\beta$	$\alpha^3\beta$	α^2	α	i	α^9	α^8	α^7	α^6	α^5	α^4	α^3
$\alpha^3\beta$	$\alpha^3\beta$	$\alpha^2\beta$	$\alpha\beta$	β	$\alpha^9\beta$	$\alpha^8\beta$	$\alpha^7\beta$	$\alpha^6\beta$	$\alpha^5\beta$	$\alpha^4\beta$	α^3	α^2	α	i	α^9	α^8	α^7	α^6	α^5	α^4
$\alpha^4\beta$	$\alpha^4\beta$	$\alpha^3\beta$	$\alpha^2\beta$	$\alpha\beta$	β	$\alpha^9\beta$	$\alpha^8\beta$	$\alpha^7\beta$	$\alpha^6\beta$	$\alpha^5\beta$	α^4	α^3	α^2	α	i	α^9	α^8	α^7	α^6	α^5
$\alpha^5\beta$	$\alpha^5\beta$	$\alpha^4\beta$	$\alpha^3\beta$	$\alpha^2\beta$	$\alpha\beta$	β	$\alpha^9\beta$	$\alpha^8\beta$	$\alpha^7\beta$	$\alpha^6\beta$	α^5	α^4	α^3	α^2	α	i	α^9	α^8	α^7	α^6
$\alpha^6\beta$	$\alpha^6\beta$	$\alpha^5\beta$	$\alpha^4\beta$	$\alpha^3\beta$	$\alpha^2\beta$	$\alpha\beta$	β	$\alpha^9\beta$	$\alpha^8\beta$	$\alpha^7\beta$	α^6	α^5	α^4	α^3	α^2	α	i	α^9	α^8	α^7
$\alpha^7\beta$	$\alpha^7\beta$	$\alpha^6\beta$	$\alpha^5\beta$	$\alpha^4\beta$	$\alpha^3\beta$	$\alpha^2\beta$	$\alpha\beta$	β	$\alpha^9\beta$	$\alpha^8\beta$	α^7	α^6	α^5	α^4	α^3	α^2	α	i	α^9	α^8
$\alpha^8\beta$	$\alpha^8\beta$	$\alpha^7\beta$	$\alpha^6\beta$	$\alpha^5\beta$	$\alpha^4\beta$	$\alpha^3\beta$	$\alpha^2\beta$	$\alpha\beta$	β	$\alpha^9\beta$	α^8	α^7	α^6	α^5	α^4	α^3	α^2	α	i	α^9
$\alpha^9\beta$	$\alpha^9\beta$	$\alpha^8\beta$	$\alpha^7\beta$	$\alpha^6\beta$	$\alpha^5\beta$	$\alpha^4\beta$	$\alpha^3\beta$	$\alpha^2\beta$	$\alpha\beta$	β	α^9	α^8	α^7	α^6	α^5	α^4	α^3	α^2	α	i

Let $H_1 \subset G$ such that $H_1 = \langle \alpha^3\beta \rangle$. Then from Table 1, $|H_1| = 2$ and the cosets of H_1 in G are $\{i,\alpha^3\beta\}$, $\{\alpha,\alpha^2\beta\}$, $\{\alpha^2,\alpha\beta\}$, $\{\alpha^3,\beta\}$, $\{\alpha^4,\alpha^9\beta\}$, $\{\alpha^5,\alpha^8\beta\}$, $\{\alpha^6,\alpha^7\beta\}$, $\{\alpha^7,\alpha^6\beta\}$, $\{\alpha^8,\alpha^5\beta\}$, $\{\alpha^9,\alpha^4\beta\}$.

Now, take $T = \{i,\alpha,\alpha^2,\alpha^3,\alpha^4,\alpha^5,\alpha^6,\alpha^7,\alpha^8,\alpha^9\}$ as a transversal and let $g = \beta$ in Lemma 3.1, then

$$\begin{aligned}
&\beta\cdot i = \alpha^{10}\beta = \alpha^7\cdot(\alpha^3\beta); &&\beta\cdot\alpha = \alpha^9\beta = \alpha^6\cdot(\alpha^3\beta);\\
&\beta\cdot\alpha^2 = \alpha^8\beta = \alpha^5\cdot(\alpha^3\beta); &&\beta\cdot\alpha^3 = \alpha^7\beta = \alpha^4\cdot(\alpha^3\beta);\\
&\beta\cdot\alpha^4 = \alpha^6\beta = \alpha^3\cdot(\alpha^3\beta); &&\beta\cdot\alpha^5 = \alpha^5\beta = \alpha^2\cdot(\alpha^3\beta);\\
&\beta\cdot\alpha^6 = \alpha^4\beta = \alpha\cdot(\alpha^3\beta); &&\beta\cdot\alpha^7 = \alpha^3\beta = i\cdot(\alpha^3\beta);\\
&\beta\cdot\alpha^8 = \alpha^2\beta = \alpha^9\cdot(\alpha^3\beta); &&\beta\cdot\alpha^9 = \alpha\beta = \alpha^8\cdot(\alpha^3\beta).
\end{aligned}$$

In this case, the transformation δ (see Lemma 3.1) is of the form

$$\delta = (1\ 8)(2\ 7)(3\ 6)(4\ 5)(9\ 10)$$

and $\beta\xi = (\alpha^3\beta)^{10} = i$, (similarly, $\alpha\xi = i$). Hence, the transfer, in this case, is a trivial homomorphism.

Remark 3.5: It is generally observed that for a 2-generator group G of order $2n$ whose elements consists of rotations α and reflections β, the following relations are valid for all $\alpha,\beta \in G$;

i) $\xi : G \to G$ defined by $\xi(\alpha^r\beta\cdot\alpha^s) = \alpha^{(r-s)\bmod n}\beta$,

ii) $\xi : G \to G$ defined by $\xi(\alpha^r\cdot\alpha^s\beta) = \alpha^{(r+s)\bmod n}\beta$,

iii) $\xi : G \to G$ defined by $\xi(\alpha^r\beta\cdot\alpha^s\beta) = \alpha^{(r-s)\bmod n}$.

In this case, the function $\zeta = \xi^{-1} : G \to G$ decomposed a transformation into multiple transformations, where

$$\zeta(\omega) = \prod_r \eta_r \quad \text{for some } \omega,\eta_r \in G.$$

Remark 3.6: It can be deduced from Table 1 that if $H = \langle \alpha \rangle$ such that $|H| = n$ and if $S = \{\alpha^r\beta\text{: } 1 \le r \le n\}$, then $H\cdot H = H$; $H\cdot S = S\cdot H = S$; $S\cdot S = H$. Hence, a transfer function ξ of H in G is restricted to these relations.

Remark 3.7: Consider a transformation $T : G \to G;\ \omega \mapsto T(\omega)$ for some $\omega \in G$. Then we can write $T_\tau(\omega) = (\omega)^\tau = \tau(\omega) = (\omega)\tau^{-1}$ for all $\tau \in G$. This operation modifies a signal in such a way that the signal is defined on the domain G and any transformation $\tau \in G$ is a mapping from G to G. Thus, the transformed signals are given by $\omega^\tau(x(n)) = \omega\tau^{-1}(x(n))$ where $x(n)$ is some discrete signal.

Lemma 3.8: Let $\xi : G \to G$ be a mapping on a finite 2-generator group G defined by $\xi_{\varphi\eta\gamma} = \xi_{\tau}$ such that $\eta\tau = i$ for some $\varphi, \eta, \gamma, \tau \in G$. Then for any discrete signal $x(n)$, $\varphi(\eta(\gamma(x(n)))) = \tau(x(n))$.

Proof: Suppose $\xi = \beta\alpha^{n}\beta^{-1}$. Since ξ is a homomorphism (Theorem 3.3), for any discrete signal $x(n)$,

$$
\begin{aligned}
\xi(x(n)) = \beta\alpha^{n}\beta^{-1}(x(n)) &= \beta(x(n))\alpha^{n}\beta^{-1}(x(n)) \\
&= \beta(x(n))\alpha^{n}(x(n))\beta^{-1}(x(n)) \\
&= \beta(x(n))(\alpha^{-n})^{-1}(x(n))\beta^{-1}(x(n)) \\
&= \beta(x(n))[(\alpha^{-n})^{-1}\beta^{-1}](x(n)) \\
&= \beta(x(n))[(\beta\alpha^{-n})^{-1}](x(n)) \\
&= \beta(x(n))[\beta(\alpha^{-n})](x(n)) \\
&= \beta(x(n))\beta(x(n))(\alpha^{-n})(x(n)) \\
&= (\alpha^{-n})(x(n)),
\end{aligned}
$$

i.e., $\xi(x(n)) = \alpha^{-n}(x(n))$.

But $\alpha^{-n}\alpha^{n} = i$. Hence, the result follows. ◊

Corollary 3.9: Suppose G is a finite 2-generator group and $\omega, \tau \in G$ where ω and τ are rotation and reflection respectively, they $\eta \in G$ can be decomposed into a product of transformations as $\eta = \tau\omega\tau^{-1}$ where $\eta = \omega^{-1}$.

Lemma 3.10: Suppose $H \subseteq G$ such that $H = \{\eta^{n} : \eta \in G\}$, a set of rotations and [G: H] = 2. Let $\eta, \tau \in G$ be arbitrary transformations (not necessarily in H). Then $H \lhd G$ and the relation $(\circ) : G \to G$ on any discrete signal $x(n)$ satisfy the following:

i. $\eta^{i} \circ \tau^{j}(x(n)) = \eta^{(i+j) \bmod n}(x(n))$ if and only if $\tau \in H$;
ii. $\eta^{i} \circ \tau^{j}(x(n)) = \tau^{(i+j) \bmod n}(x(n))$ if and only if $\tau \notin H$;
iii. $\tau^{i} \circ \eta^{j}(x(n)) = \eta^{(i-j) \bmod n}(x(n))$ if and only if $\tau \notin H$;
iv. $\mu^{i} \circ \tau^{j}(x(n)) = \eta^{(i-j) \bmod n}(x(n))$ if and only if $\mu, \tau \notin H$.

In particular, since H is Abelian, $(\eta \circ \rho)(x(n)) = \eta(x(n)) \circ \rho(x(n))$ for all $\eta, \rho \in H$ where $\circ$ denote the product of permutation.

3.2 Discrete-time signal

In this section, it is assumed that elements of G are of the form $x(n)$. Thus, the operator $L(G)$ can be defined as an operator over a signal space.

We define two special functions as follows:

i. A δ-function $\delta(n)$ defined by $\delta(n) = \begin{cases} 1 & if \quad n = 0, \\ 0 & if \quad n \neq 0. \end{cases}$

ii. A step function $\mu(n)$ defined by $\mu(n) = \begin{cases} 1 & if \quad n \geq 0, \\ 0 & if \quad n < 0. \end{cases}$.

Now if $n = 0$, then $\mu(n) = \delta(n)$, if $n = 1$, then $\mu(n) = \delta(n - 1)$ and so on. Thus,

$\mu(n) = \delta(n) + \delta(n - 1) + \delta(n - 2) + \ldots$

$$\text{i.e. } \mu(n) = \sum_{k=0}^{\infty} \delta(n-k). \tag{3.1}$$

Again, since $\delta(n) = 0$ for $n < 0$,

$$\mu(n) = \sum_{k=-\infty}^{n} \delta(k). \tag{3.2}$$

But $\mu(n) = \delta(n)$ if and only if $n = 0$. Thus from Equations 3.1 and 3.2, it follows that

$$\mu(n) = \sum_{k=-\infty}^{\infty} \delta(n-k). \tag{3.3}$$

Now, $T(x(n)) = y(n)$ for any transformation T on some $x(n) \in G$ which produces an output $y(n)$. Thus by linearity,

$L(\alpha x(n)) = \alpha L(x(n))$;
$L(\alpha x(n) + \beta x(n)) = \alpha L(x(n)) + \beta L(x(n))$,

where the coefficients α and β are constants. Hence, by Equation 3.3, any discrete signal $x(n)$ can be express as

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k), \tag{3.4}$$

where the coefficients $x(k)$ can be real or complex. Hence, by Equation 3.4, any signal $x(n)$ can be represented as a linear superposition of weighted and shifted impulse response (signals), also called signal basis.

Again, if

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k),$$

then

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)L(\delta(n-k)). \tag{3.5}$$

Moreover, by linear time-invariant, if $h(n)$ is any discrete time signal such that $L(x(n)) = h(n)$, then $L(x(n\text{-}k)) = h(n\text{-}k)$. Thus if $L(\delta(n)) = h(n)$, then $L(\delta(n\text{-}k)) = h(n\text{-}k)$. Hence, from Equation 3.5,

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

$$= x(n) * h(n).$$

Thus, $y(n) \in G$ is expressed as a product of convolution $x(n) * h(n)$ for some $x(n), h(n) \in G$.

Example 3.2.1: Let $x(-3n + 2)$ be any discrete-time signal and let δ, σ and μ be the transformations shifting, flipping and scaling respectively. Then

$\delta(n) = x(n + 2)$;

$\sigma(n) = \delta(-n)$;

$\mu(n) = \sigma(3n) = \delta(-3n) = x(-3n + 2)$.

Thus, $x(-3n + 2)$ is decomposed by the transformations δ, σ and μ such that

$x(-3n + 2) = \delta(n) + \sigma(n) + \mu(n)$.

4 Summary

The study has been carried out only on finite groups and discrete signals for demonstration purposes. It shows that apart from the use of character functions, which has been extensively examined in the group theoretical concept of signal processing, other transformation groups also provide lots of opportunities for newer and more efficient approaches to signal processing which has not yet been realized in the current literature. Therefore, it reveals more options for signal processing using group theoretical tools. The results showed that many transformations can be decomposed into multiple transformations over a discrete-time signal which can be applied in areas like frequency analysis, feature extraction, edge detection and object identification.

5 Conclusion

According to group representation theory, new representations can be generated by direct product (or tensor product) of any two representations of a group. In this case, their irreducible representations will also be a direct product. This work presents a decomposition of a single transformation into multiple transformations which can be used for signal processing. Thus, a different approach to signal processing with the choice of a group of transformations is established.

Competing Interests

Authors have declared that no competing interests exist.

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