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# Bayesian Estimation and Economic Analysis of Under-Replicated Field Trials With a Linear Response Plateau Function

Whoi Cho[1], Dayton M. Lambert[1], Alimamy Fornah[2] & William R. Raun[3]

[1] Department of Agricultural Economics, Oklahoma State University, Stillwater, USA

[2] Department of Agriculture, Biology, and Health Sciences, Cameron University, Lawton, USA

[3] Department of Plant and Soil Sciences, Oklahoma State University, Stillwater, USA

Correspondence: Dayton M. Lambert, Department of Agricultural Economics, Oklahoma State University, Stillwater, OK 74078, USA. Tel: 1-405-744-7465. E-mail: dayton.lambert@okstate.edu

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**Abstract**

The linear response with plateau (LRP) is widely used in agronomic and agricultural economic studies of crop yield response. This empirical example uses data from an under-replicated experiment to compare maize (*Zea maize* L.) yield response to nitrogen under different plant and corridor row spacing. Not all replications received a 0-nitrogen rate, making estimation of the LRP difficult because data for the intercept terms is absent. We leverage information from other treatments using Bayesian methods to estimate the yield response of each treatment using a LRP function, given limited replication and absence of check plots for some treatments. We use a linearized LRP, which bypasses using the "min" operator typically required to estimate LRP functions. Economically optimal nitrogen rates were determined and net returns from treatments compared from the perspective of risk-averse producers. The wide plant/narrow row treatment was most profitable when the decision rule was to apply nitrogen. The statistical procedure used here may be useful for exploratory analyses of pilot agronomic trials that may include unbalanced and under-replicated treatments.

**Keywords:** maize, plant and row spacing, linear response plateau, Bayesian Markov Chain Monte Carlo, nitrogen

## 1. Introduction

The linear response with plateau (LRP) model has broad theoretical and practical appeal to both agronomists and economists' study of crop response to inputs. The LRP is based on von Liebig's law of the minimum, which states:

> *The crops on a field diminish or increase in exact proportion to the diminution of increase of mineral substances conveyed to it in manure…by deficiency or absence of one necessary constituent, all the others being present, the soil is rendered barren for all those crops to the life of which that one constituent is indispensable* (cited in Dillon & Anderson, 1990, p. 81).

In other words, plant growth occurs at a constant rate with nutrients contributing to its production in fixed proportions until some factor becomes limiting (Blackman, 1905; Swanson, 1963). A horizontal plateau over a range of input levels depicts this situation (Grimm et al., 1987). From an agronomic perspective, with good prior information about growing conditions, soil type, and plant growth, Linear Response Plateau (LRP) models require only three input levels for identification; check plots and two additional application rates (Babcock et al., 1996).

Numerous agronomic studies have used the LRP to estimate plant response to fertilizer inputs and to determine corresponding biological and/or economically optimal application rates. Dillon and Anderson (1990) cite 17 studies in their review of the LRP. Mangiafico and Guillard (2005) determined optimal nitrogen applications for turf grass using the LRP function. Tembo et al. (2008) used a stochastic version of the LRP to estimate optimal nitrogen fertilizer rates for wheat over multiple growing seasons. Haque et al. (2009) estimated LRP functions to identify nutrient-efficient perennial grass species and optimal nitrogen application rates. Tumusiime et al. (2011) used a stochastic LRP to determine optimal nitrogen rates for rye and ryegrass. Boyer et al. (2012) used a LRP to determine profit-maximizing nitrogen rates for switchgrass, while Haankuku et al. (2014) used the LRP function to estimate forage sorghum yield response to nitrogen. Harmon et al. (2016) used LRP functions to determine economically optimal potassium fertilizer rates for cotton. Dhakal et al. (2019) used a LRP to estimate carryover dynamics of nitrogen on cotton yield response.

We estimate maize yield response to nitrogen planted at different row widths with a LRP using Bayesian procedures. Estimation of the LRP with Bayesian methods is not new. Holloway and Paris (2002) were first to apply Bayesian procedures to estimate the LRP. Ouedrago and Brorsen (2018) estimated a stochastic plateau function with a hierarchical Bayesian model to determine the optimal nitrogen rates for winter wheat. The advantage of using Bayesian methodology in this application is that prior information on treatment yields can address problems arising from under-replication and unbalanced experimental designs. The experimental data used in this application was from a randomized complete block design with three nitrogen applications of four treatments, but it was unbalanced and under-replicated. Two of the four treatments did not receive 0-nitrogen check strips. Without check plot repetitions, it is difficult to estimate an intercept, which is essential for identifying the LRP's slope and plateau parameters using nonlinear least squares or maximum likelihood (ML). We estimate the LRP parameters by using response information from other treatments as Bayesian priors.

Our example concludes with an economic application using the posterior estimates of the LRP. Economically optimal nitrogen fertilizer rates for different plant/row spacing combinations for maize (*Zea maize* L.) are evaluated using the posterior distributions of the model's intercept, slope, and plateau coefficients. Net returns generated under different plant/row spacing treatments and nitrogen fertilizer rates are compared in a partial budget analysis assuming 1) a risk-neutral producer who maximizes expected profit and 2) a producer who prefers upside variability in net returns but is averse to downside risk. Stochastic dominance is used to evaluate the second case.

## 2. Method

### *2.1 Linear Response With Plateau: Background and Extension*

Paris and Knapp (1989), Dillon and Anderson (1990), and Paris (1992) review the linear response plateau's (LRP) history in agronomic and economic studies [Note 1]. The LRP depicts the biological effects of limiting factors on plant growth as:

$$y = \min\{f_1(x_1), f_2(x_2) \ldots f_K(x_K), M\} \tag{1}$$

where, $y$ is crop or animal production, $x_j$ a vector of inputs (*e.g.*, $j$ = nitrogen, phosphorous, or potassium), and $M$ a plateau common to all inputs. The plateau is the highest obtainable biological maximum yield or production, subject to a limiting factor or input. The functions $f_j(\cdot)$ may include different inputs and can be linear or nonlinear functions such as the quadratic, Spillman, or Mitscherlich-Baule forms (Paris, 1992). Linear, single input functions were most commonly used in the majority of the studies reviewed here.

Assuming a linear relationship between crop response and inputs, then for each factor $j = 1, 2, \ldots k$, yield response ($y$) is:

$$y = \min\{\alpha_1 + \beta_1 \cdot x_1, \alpha_2 + \beta_2 \cdot x_2 \ldots \alpha_K + \beta_K \cdot x_K, M\} \tag{2}$$

where the $\alpha$'s are positive intercept terms and $M$ is a yield plateau common to all response regimes. The marginal physical product of plant growth with respect to input $j$ is the positive slope parameter $\beta_j$. Writing Equation 2 as a two-part model provides economic intuition and is the starting point for linearizing the LRP. For each input determining production or yield response, the two-part specification is:

$$y = \begin{cases} \alpha_j + \beta_j \cdot x_j \text{ for } 0 \le x_j < x_j^* \\ \alpha_j + \beta_j \cdot x_j^* \text{ for } x_j \ge x_j^* \end{cases} \tag{3}$$

where, $x_j^*$ is the critical value of input $x_j$ and calculated as $x_j^* = [M - \alpha_j]/\beta_j$. The critical input level for the $j$th input corresponds with a "knot" connecting the intercept to a plateau with a line ascending at rate $\beta_j$, the linear response. Formulated this way, response to an input factor increases positively until that input becomes limiting, where after the marginal physical product is zero and the plateau $M^* = \alpha_j + \beta_j \cdot x_j^*$ is obtained (Anderson & Dillon, 1990). When inputs are purchased and there is a market for the output, the two-part formulation implies that a factor should be used if the ratio of the per unit cost of input $j(r_j)$ to the per unit price received for the commodity (p) is less than the marginal physical product; *i.e.*, the producer should use input $x_j^*$ just until $r_j \le p \cdot \beta_j$.

The LRP model's constants must be estimated when the relationship between yield or production and inputs is imperfect. The error term of Equation 2 are typically assumed to be linear-additive, *i.e.*,

$y = \min\{f_1(x_1), \ldots f_K(x_K), M\} + u$ [Note 2]. Earlier applications estimating this particular specification used a dummy variable approach to approximate the join point location of the plateau and linear response with multiple step-wise linear regressions (for example, Paris & Knapp, 1989). With the advent of better solver algorithms, conventional maximum likelihood or Bayesian MCMC estimation techniques can be used to estimate the LRP parameters, with the "min" function entering directly into a likelihood function (Holloway & Paris, 2002). In small sample settings, the nonlinear least squares, ML, and MCMC estimators of the LRP with linear-additive errors produce similar results (Brorsen, 2013) [Note 3].

The discontinuous and non-differentiable nature of the LRP once posed computational before advances in solver optimization routines (Q. Paris & P. Paris, 1985; Paris & Knapp, 1989). This research uses an alternative specification of the LRP model that bypasses use of the "min" function and allows for additional flexibility in parameterization of the response model. Not surprisingly, the linearized LRP intercept, slope, and plateau estimators are identical to those obtained using conventional likelihood methods whereby the "min" function enters directly into a log likelihood or least squares minimization function. We reformulate the LRP with linear-additive errors by endogenizing the optimal input levels, $x_j^*$, in the linear part of the model. Consider first the case where a single input is used. Let the indicator function $I(x < x^*) = 1$ when the argument is true, 0 otherwise. Yield response follows as:

$$y = I(x \le x^*) \cdot (\alpha + \beta \cdot x) + [1 - I(x \le x^*)] \cdot (\alpha + \beta \cdot x^*) + u \tag{4}$$

which is algebraically equivalent to Equations 2 and 3. The latent indicator variables sort observed yield-input pairs into linear response or plateau domains, subject to an applied input level or rate. This specification is similar to threshold autoregressive models appearing in the time series literature (Luukkonen et al., 1988; Lundbergh et al., 2003). This parametrization is also similar to Paris and Knapp's (1989) iterative stepwise least squares approach that uses dummy variables to identify join points, but Equation 4 suggests the indicator variables enter directly into the response function as nonlinear, discontinuous functions of the linear response and plateau parameters. With linear-additive errors, the data generating process for the LRP is:

$$y = \begin{Bmatrix} \alpha + \beta \cdot x \colon I(x \le x^*) \\ \alpha + \beta \cdot x^* \colon I(x > x^*) \end{Bmatrix} + u \tag{5}$$

Evaluated at the biologically optimal input rate, yield is:

$$y = \begin{Bmatrix} \alpha + \beta \cdot x \colon I(x \le x^*) \\ M \colon I(x > x^*) \end{Bmatrix} + u \tag{6}$$

where, $u$ is the error term with $\mathrm{E}(u) = 0$ and $\mathrm{Var}(u) = \sigma^2$.

The parameters maximizing this function can be estimated directly using maximum likelihood, nonlinear least squares, or Bayesian Markov Chain Monte Carlo (MCMC) procedures [Note 4]. One advantage of using a Bayesian MCMC procedure is that the posterior distribution of the parameters can be used directly in *ex-post* partial budget analyses to derive empirical distributions of yields, optimal fertilizer application rates, and net returns under competing management scenarios. The empirical distributions can be resulted from a converged chain. They can also be used directly to compare management scenarios statistically under the assumption that producers maximize expected net returns (the risk-neutral case) or that producers are averse to variability in net returns (risk-averse). In terms of limited experimental replication, another advantage is that Bayesian estimates are valid regardless of sample size or degrees of freedom (McElreath, 2015).

### *2.2 Empirical Application*

The empirical application estimates maize response to nitrogen sown in different plant and row widths. Plant and row spacing has long been understood to affect maize yield and the amount of nitrogen fertilizer required. Optimal row and plant spacing of maize reduces weed competition for nutrients and sunlight and increases maize productivity and nitrogen use efficiency (Widdicombe & Thelen, 2002; Worku & Astatkie, 2011; Mattera et al., 2013; Testa et al., 2016). Yields for narrow row maize are generally higher than wider row spacing (Widdicombe & Thelen, 2002; Lambert & Lowenberg-DeBoer, 2003; De Bruin & Pedersen, 2008; Mohammadi et al., 2012). Muldoon and Daynard (1981) found that variability in intra- row spacing affects maize yield. When maize was planted uniformly, yields increased and plant-to-plant variation in size was reduced (Martin et al., 2005; Rossini et al., 2011).

Inter- and intra-row widths affect optimal nitrogen fertilizer rates for maize (Martin et al., 2005; Barbieri et al., 2008; Boomsma et al., 2009; Srivastava et al., 2018). Applying too much nitrogen fertilizer diminishes nitrogen use efficiency and may cause soil acidification (Chen et al., 2014; Zhu et al., 2016). Some research finds that different row and plant spacing require different nitrogen fertilizer rates (Karlen & Camp, 1985; Lewis & Knight, 1987). Other studies conclude that grain yield response to applied nitrogen and the optimal nitrogen application rates are similar regardless of plant and row spacing (Shapiro & Wortmann, 2006). Barbieri et al. (2008) showed that optimal nitrogen fertilizer rates were lower for maize planted in narrower rows.

The empirical application uses data from a two-year field experiment (2017 and 2018) managed by the Plant and Soil Science Department of Oklahoma State University. The objective of the experiment was to identify the optimum plant-to-plant spacing and the optimum row spacing for maize and response to nitrogen. Maize plots were located eight miles west of Stillwater, OK [Note 5]. The research plots were on Pulaski fine-sandy loam soil (coarse/loamy, mixed non-acid, thermic, Typic, and Ustifluvent) (USDA/NRCS soil taxonomy). The experiment was a randomized complete block design with three nitrogen applications of four treatments, each replicated three times.

Plot sizes were 9.94-ft × 19.88-ft. Urea fertilizer (46-0-0) was applied as side-dress at 0, 54, and 107 lb ac$^{-1}$. Maize was planted 1.97 inches deep for all treatments. Between-plant spacing treatments were 0.49-ft and 0.98-ft. Row spacing treatments were 1.67-ft and 2.49-ft. Plant-to-plant spacing was maintained by marking a string at 5.91 inches or 11.81 inches, according to the treatment structure. Wide plant/narrow row spacing under Treatment 3 (0.98-ft and 1.67-ft) and wide plant/wide row spacing under Treatment 4 (0.98-ft and 2.49-ft) did not receive the 0-nitrogen rate treatment for logistical and budgetary reasons (Table 1).

Table 1. Maize yield and plant-row spacing treatments

| Treatment | ----- Treatment 1 ----- | | | ----- Treatment 2 ----- | | | --- Treatment 3 --- | | --- Treatment 4 --- | |
|---|---|---|---|---|---|---|---|---|---|---|
| Nitrogen rate (lbs ac$^{-1}$) | 0 | 54 | 107 | 0 | 54 | 107 | 54 | 107 | 54 | 107 |
| Mean yield (bu ac$^{-1}$) | 56 | 87 | 118 | 38 | 63 | 89 | 98 | 155 | 73 | 99 |
| Standard error | 18 | 10 | 49 | 13 | 9 | 40 | 15 | 71 | 14 | 39 |
| Minimum yield (bu ac$^{-1}$) | 38 | 70 | 61 | 25 | 49 | 38 | 73 | 79 | 47 | 57 |
| Maximum yield (bu ac$^{-1}$) | 86 | 106 | 186 | 58 | 75 | 134 | 115 | 230 | 98 | 144 |
| *n* | 6 | 12 | 6 | 6 | 12 | 6 | 12 | 6 | 12 | 6 |
| *N* | 24 | | | 24 | | | 18 | | 18 | |

*Note*. Treatments indicated (plant; row) spacing: Treatment 1 is (0.49-ft; 1.67-ft), Treatment 2 is (0.49-ft; 2.49-ft), Treatment 3 is (0.98-ft; 1.67-ft), and Treatment 4 (0.98-ft; 2.49-ft).

At harvest, maize samples were collected and the grain weighed. Grain weight was converted to yield using Raun et al. (2002)'s methodology. The wider plant spacing generally produced higher yields than the narrower plant spacing treatment. The average yield for the wide plant and narrow row spacing treatment (Treatment 3; 0.98-ft and 1.67-ft) was 155 bu ac$^{-1}$ at 107 lbs ac$^{-1}$ nitrogen fertilizer (Table 1). The narrow plant and wide row spacing of Treatment 2 (0.49-ft and 2.49-ft) yielded 89 bushels at the highest nitrogen fertilizer application. For Treatments 1 and 2, the 0-nitrogen treatments yielded 56 and 118 bu ac$^{-1}$, respectively.

*2.3 Estimation*

We estimate maize yield response to nitrogen fertilizer planted in different plant/row spacing as:

$$y_{ik}= \sum_{k=1}^{4} d_k \cdot [I^k\left(N_{ik} \leq N_{ik}^*\right)\left(\beta_{0k} + \beta_{1k} \cdot N_{ik}\right) + I^k\left(N_{ik} > N_{ik}^*\right)\left(\beta_{0k} + \beta_{1k} \cdot N_{ik}^*\right)] + u_{ik} \quad (7)$$

where, $y_{ik}$ is maize yield in treatment replicate $i$; $k$ indexes treatments (plant space/row space), 1 = (0.49-ft, 1.67-ft), 2 = (0.49-ft, 2.49-ft), 3 = (0.98-ft, 1.67-ft), and 4 = (0.98-ft, 2.49-ft); $N_{ik}^*$ is the critical value of nitrogen fertilizer rate $N_{ik}$; $\beta_{0k}$ and $\beta_{1k}$ are positive intercept and slope parameters; I is an indicator variable equal to 1 when applied nitrogen is less than or equal to the nitrogen rate corresponding with the yield plateau (0 otherwise); $d_k$ is a (0, 1) variable indicating an observed yield from treatment $k$; and $u_{ik}$ is an independent and identically distributed random error with an expected value of zero and a constant variance. The variance term is constant across treatment yields under the assumption that random variation due to weather and other stochastic events occurred with equal likelihood across treatment plots.

Yield intercept, linear, and plateau parameters were estimated using SAS's PROC MCMC, a Bayesian MCMC routine (SAS, 2014). We use this procedure for two reasons. First, the sample size of each treatment is relatively small, with $N$ = 24 for treatments 1 and 2 and $N$ = 18 for treatments 3 and 4 (Table 1). There were 24 observations for treatments 1 and 2, both with $n$ = 6 replications for the check plot and highest nitrogen rate, and $n = 12$ observations for intermediate fertilizer rate of 54 lbs ac$^{-1}$. Second, treatments 3 and 4 did not receive the 0-nitrogen rate, rendering estimation of an intercept term for these treatments exceedingly difficult if standard maximum likelihood or nonlinear least squares procedures were used to estimate the LRP parameters.

We leverage yield information from the check plots of treatments 1 and 2 as priors for the intercept terms of treatments 3 and 4. An advantage of Bayesian MCMC estimators is that they are valid regardless of sample size (Brorsen, 2013; McElreath, 2015). Even so, the power of a Bayesian estimator critically depends on prior information on a parameter's distribution. We compare the performance of models under different priors with the deviance information criterion (DIC). Preferred models are those with smaller DICs (Spiegelhalter et al., 2002). The DIC is calculated with the posterior densities of the respective models.

Model specifications under different prior assumptions are summarized in Table 2. For all models, a diffuse prior was assumed for the error variance parameter. The prior distribution for the variance term is the inverted Gamma distribution (IG) with a shape parameter of 0.001 and a scale parameter of 1,000. For the other parameters, we use informative priors based on the univariate statistics of the observed treatment yields.

The first model (Model 1) assumes the intercept, slope, and plateau parameters are Gaussian (normal) distributed. Model 1's specification is:

$$y_{ik} \sim G\left([I^k\left(N_{ik} \le N_{ik}^*\right)\left(\beta_{0\mathrm{k}} + \beta_{1k} \cdot N_{ik}\right) + I^k\left(N_{ik} > N_{ik}^*\right)\left(\beta_{0k} + \beta_{1k} \cdot N_{\mathrm{ik}}^*\right)], \sigma^2\right) \tag{8}$$

$$\pi\left(\beta_{0k}\right) \sim G(41, 7.25) \;\; \forall k \tag{9}$$

$$\pi\left(\beta_{1k}\right) \sim G(0.75, 0.10) \;\; \forall k \tag{10}$$

$$\pi(M_k) \sim G(150, 4.34) \;\; \forall k \tag{11}$$

$$\pi(\sigma^2) \sim IG(0.001, 1000) \tag{12}$$

where, $G$ denotes the Gaussian (standard normal) distribution. We used the means and standard deviations of the intercept, slope, and plateau estimates reported by Boyer et al. (2013) in their analysis of corn yield response to nitrogen as a reference distribution for model 1. For all plant/row spacing treatments evaluated with model 1, the mean (standard deviation) of the intercept priors was 41 bu ac$^{-1}$ (7.25), and the mean (standard deviation) of the linear response priors were 0.75 bu lb$^{-1}$ of applied nitrogen (0.10) (Table 2). The Gaussian prior for the plateau parameters was 150 bu ac$^{-1}$, with a standard deviation of 4.34.

Table 2. Priors for models evaluated and deviance information criterion

| Model | Intercept | N rate Slope | Plateau yield | Error variance | DIC [a] |
|---|---|---|---|---|---|
| 1 | $G(41, 7.25)$ | $G(0.75, 0.10)$ | $G(150, 4.34)$ | $IG(0.001, 1000)$ | 807 |
| 2 | $Uni(\min_0, \max_0)$ | $Uni(0,1)$ | $Uni(\min_{107k}, \max_{107k})$ | $IG(0.001, 1000)$ | 811 |
| 3 | $Uni(\min_0, \max_0)$ | $Uni(0,5)$ | $Uni(\min_{107k}, \max_{107k})$ | $IG(0.001, 1000)$ | 788 |
| 4 | $Uni(\min_0, \max_0)$ | Half-Cauchy | $Uni(\min_{107k}, \max_{107k})$ | $IG(0.001, 1000)$ | 809 |

*Note*. $\min_0$ and $\max_0$ are the minimum and maximum yield of data when no nitrogen fertilizer applied, and $\min_{107k}$ and $\max_{107k}$ are the minimum and maximum yield of data when 107 lb ac$^{-1}$ nitrogen fertilizer applied by each treatment $k = 1, \ldots 4$.

[a] DIC, Deviance Information Criterion.

Models 2, 3 and 4 used Equation 8 as the yield response prior and Equation 12 as the prior for the error variance. We evaluated alternative priors for the intercept, slope, and plateau parameters for models 2, 3, and 4. A lower bound of zero was used for the treatment intercept terms. Uniform distribution priors may be preferable when information on individuals, treatments, or group characteristics is limited (Gelman, 2006). Our experimental data is relatively limited, with treatments 3 and 4 receiving no check strips. Naturally, one would expect yields to be zero or positive even in the absence of fertilizer. The priors for the treatment intercepts of models 2, 3, and 4 were the uniform distribution (*Uni*), specified as $\pi\left(\beta_{0k}\right) \sim Uni(\min_0, \max_0)$ for all treatments. The $\min_0(\max_0)$ terms are the lowest (highest) yield observed in treatments 1 and 2 (Table 1).

The uniform distribution was also used as plateau priors for Models 2, 3, and 4. All treatments received the maximum nitrogen fertilizer of 107 lb ac$^{-1}$. The corresponding yields observed under the maximum applied fertilizer rate for each plant-row spacing treatment served as the lower and upper bounds of the plateau priors as $\pi(M_k) \sim Uni(\min_{107k}, \max_{107k})$ (Table 1).

We evaluated the performance of models 2, 3, and 4 assuming different priors for the LRP slope parameters (Table 2). The expectation is that the slope parameters are either positive and upward sloping or flat, indicating that maize exhibits no response to nitrogen. Models 2 and 3 assume a uniform distribution for the slope parameter; $\pi(\beta_{1k}) \sim Uni(0,\ b)\ \forall k$, with the upper bound set to $b = 1$ for model 2 and $b = 5$ for model 3. For model 2, the 0 to 1 range includes values of slope estimates from previous literature (*i.e.*, Boyer et al., 2013). For model 3, the upper bound of 5 admits an extreme case where maize is highly responsive to nitrogen with yield plateauing quickly at low amounts of N fertilizer. For model 4, the prior used for the slope parameters of each treatment was the Half-Cauchy distribution. Thus, model 4 also includes the case where maize is unresponsive to nitrogen fertilizer or positive/upward sloping (Table 2). The Half-Cauchy distribution was defined as a $t$ distribution truncated at zero with 3 degrees of freedom; *i.e.*, $\pi(\beta_{1k}) \sim t_{trunc}(0,\ 3)\ \forall k$ (McElreath, 2015). This prior has thicker tails and accommodates outliers (Gelman et al., 2013).

Model convergence was validated using the effective sample sizes (ESS) of the posterior distributions of each parameter and the remaining autocorrelation between draws of the Markov chains. Relatively small ESS values and high autocorrelation are indicative of convergence problems (Kass et al., 1998). As a parameter's ESS approaches the number of posterior draws used to populate the chain, one can conclude that the model exhibits good mixing properties size (Che & Xu, 2010). The relative efficiency of each parameter's chain is calculated as the parameter's ESS divided by the number of Monte Carlo (MC) draws. Efficiency scores close to 100% indicate chain convergence. We sampled 1,000 MC draws from parallel chains, with a burn-in of 50,000 and a thinning value of 1,000. Thus, the total number of draws is 1,050,000. We reported the highest posterior density (HPD), or the modes, of the estimated parameters and their corresponding HPD 90-percent credible intervals (HDPI).

A Kolmogorov-Smirnov (KS) two-sample test is used to compare statistically the posterior distributions of the LRP intercept, slope, and plateaus of each treatment. The KS test is a non-parametric test of the equality of two distributions [Note 6]. The null hypothesis is that pairs of distributions originate from the same reference distribution.

### 2.3 Economic Analysis

For each treatment, we generate economically optimal nitrogen rates, yields, and profit using the posterior empirical distributions of the parameter estimates. The risk-neutral decision maker solves the following problem to maximize expected profit:

$$\max_{\substack{N_k^* \\ k \in \{1,2,3,4\}}} \mathrm{E}(\pi_k) = \begin{cases} p \cdot \hat{\beta}_{0k} - F_k^C \text{ if } \frac{r}{p} \geq \hat{\beta}_{1k} \\ p \cdot \hat{M}_k - r \cdot N_k^* - F_k^C \text{ if } \frac{r}{p} < \hat{\beta}_{1k} \end{cases} \tag{13}$$

where, $\mathrm{E}(\pi_k)$ is the expected profit under treatment k evaluated at maize price $p$; "ˆ" indicates a parameter estimate drawn from a posterior distribution; $F_k^C$ are the per acre fixed costs of each treatment; and r are fertilizer nitrogen costs (\$ lb$^{-1}$). The price of maize is \$3.40 bu$^{-1}$, based on the USDA National Agricultural Statistics Service (NASS) annual reports for Oklahoma (USDA, 2018). The price of nitrogen is \$0.43 lb$^{-1}$, and was calculated from a urea fertilizer price of \$0.20 lb$^{-1}$ (Farmers Coop Association of Snyder, 2019). If the marginal value product of fertilizer exceeds its per unit cost, then the producer applies nitrogen at the optimal rate, $N_k^*$. Each treatment uses a different number of seeds per acre because of the different spacing. The technology costs are reflected in the change in plant population density according to plant width and row spacing (Treatment 1, 52,922 seeds ac$^{-1}$; Treatment 2, 35,514 seeds ac$^{-1}$; Treatment 3, 26,461 seeds ac$^{-1}$; Treatment 4, 17,757 seeds ac$^{-1}$). The price of maize seed is \$3.20 per one million count (Plastina, 2019). Seed costs for each treatment are therefore \$0.17, \$0.11, \$0.08, and \$0.06 per acre, respectively. Decision makers who prefer more to less but are unconcerned about the variability or skewness of returns are risk-neutral. For these individuals, the plant/row spacing treatment that generates the highest expected return would be the preferred technology.

### 2.4 Stochastic Dominance Analysis

Stochastic dominance is used to generate a risk-preferred ordering of the plant width/row spacing treatments. Stochastic dominance maintains two key assumptions about human behavior (Lambert & Lowenberg-DeBoer, 2003). First, most decision makers prefer more to less. Second, most people are averse to low value outcomes. The second statement suggests that individuals are risk-averse, but this does not necessarily mean that risk-averse individuals prefer to avoid gambles with highly variable returns. The same decision maker might prefer technologies that generate, with some positive probability, right-skewed net returns or yields.

Two decision rules resulting from the assumptions above are first-degree stochastic dominance (FDSD) and second-degree stochastic dominance (SDSD) criteria (Anderson et al., 1977). First-degree stochastic dominance assumes that persons generally prefer technology options that generate higher returns over ones that yield lower returns. For example, when technology A dominates B by the first-degree rule, the net returns from *A* are higher than B at every probability in the empirical distribution. Graphically, for net returns this means that the empirical distribution of A is always to the right of B.

The SDSD criterion assumes decision makers are risk-averse. Mathematically, the SDSD criterion is $\int_{-\infty}^{z} F_A(\pi)\mathrm{d}z \leq \int_{-\infty}^{z} F_B(\pi)\mathrm{d}z$ for all net returns over support $z$ with at least one strict inequality (Anderson & Dillon, 1992). In other words, SDSD enumerates the area between the crossover points of two competing distributions at every level of probability in their respective empirical distributions of net returns. As a general rule, if the lowest value in the net return distribution of *A* is smaller than the minimum net return value of *B*'s distribution, then technology *A* can never dominate B even if net returns from *A* are higher than *B*'s at every point in the distribution thereafter. Note that FDSD implies SDSD, but not the converse. Stochastic dominance comparisons were conducted using Simetar® software (Richardson et al., 2006) [Note 7].

## 3. Results

Discussion of maize yield response to nitrogen under the different plant-row spacing treatment uses the results of model 3 because this model had the smallest deviance information criterion (DIC = 788) (Table 2). Even so, the power of a Bayesian estimator critically depends on prior information on a parameter's distribution [Note 8]. Inspection of the posterior distributions indicates a low correlation between draws, relatively high effective sample sizes, and overall parameter convergence (Figure 1). The average efficiency score was 0.79, with a low (high) efficiency of 0.44 (1.28) (Table 3). The intercept and the error variance parameters exhibit good mixing properties, while the ESS of the slope estimates exhibited the largest HDPI range.

Table 3. Parameter efficiency for model 3

| Parameter | Treatment 1 | Treatment 2 | Treatment 3 | Treatment 4 |
|---|---|---|---|---|
| Intercept | 0.93 | 1.10 | 0.91 | 0.87 |
| N rate slope | 0.44 | 0.64 | 1.28 | 0.54 |
| Yield plateau | 0.54 | 0.53 | 0.94 | 0.52 |
| Error variance | 0.97 | | | |
| Average | 0.79 | | | |

*Note*. Efficiency is calculated as the effective sample size divided by the number of Monte Carlo iterations (1,000). Treatments indicated (plant-row) spacing: Treatment 1 is (0.49-ft; 1.67-ft), Treatment 2 is (0.49-ft; 2.49-ft), Treatment 3 is (0.98-ft; 1.67-ft), and Treatment 4 is (0.98-ft; 2.49-ft).

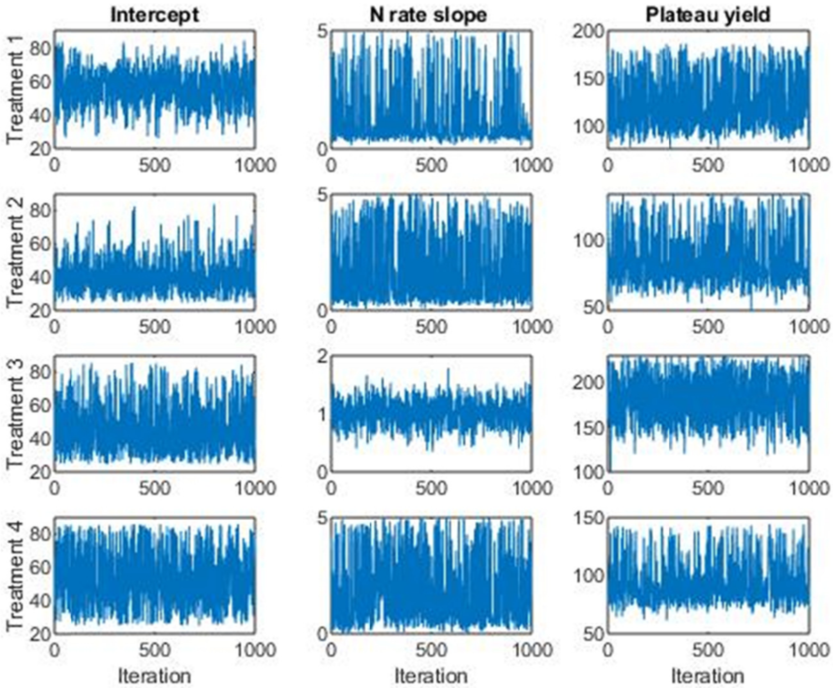


Figure 1. Posterior parameter estimate chains for model 3

Figure 2 graphically depicts the LRP functions for each treatment (evaluated at posterior distribution modes) in addition to the other 999 simulated response functions. Grey lines are the simulated maize yield estimates for intercept, linear response, and plateau. Red lines indicate the predicted yield response at the mode (highest posterior density) of the distributions. The black dots denote observed yield. The wide plant/narrow row spacing (Treatment 3; 0.98-ft/1.67-ft plant/row spacing) had the highest plateau yield of 160 bu ac$^{-1}$ and an optimal nitrogen fertilizer rate of 125 lb ac$^{-1}$ (Table 4). The narrow plant/wide row spacing treatment (Treatment 2; 0.49-ft plant spacing and 2.49-ft row spacing) generated the lowest yield (72 bu ac$^{-1}$) at 68 lb ac$^{-1}$ of nitrogen fertilizer (Figure 3).

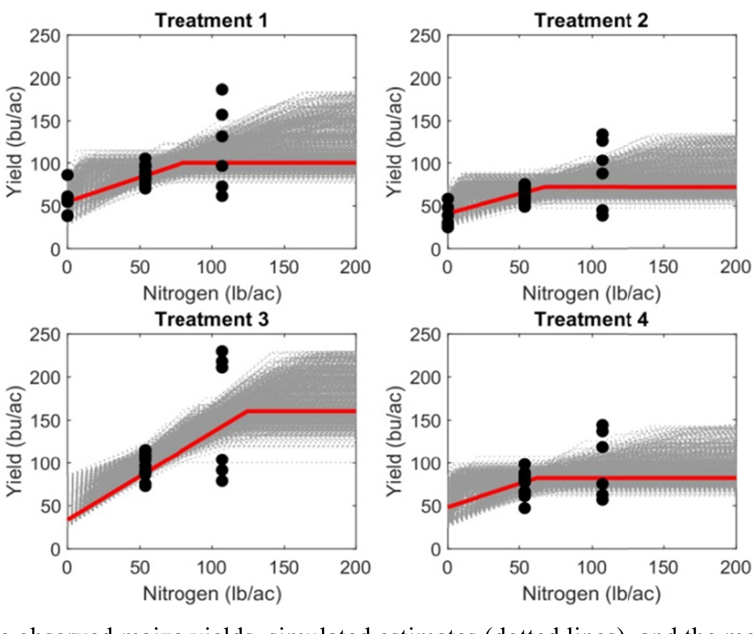


Figure 2. Plots of the observed maize yields, simulated estimates (dotted lines), and the median predicted yield from model 3

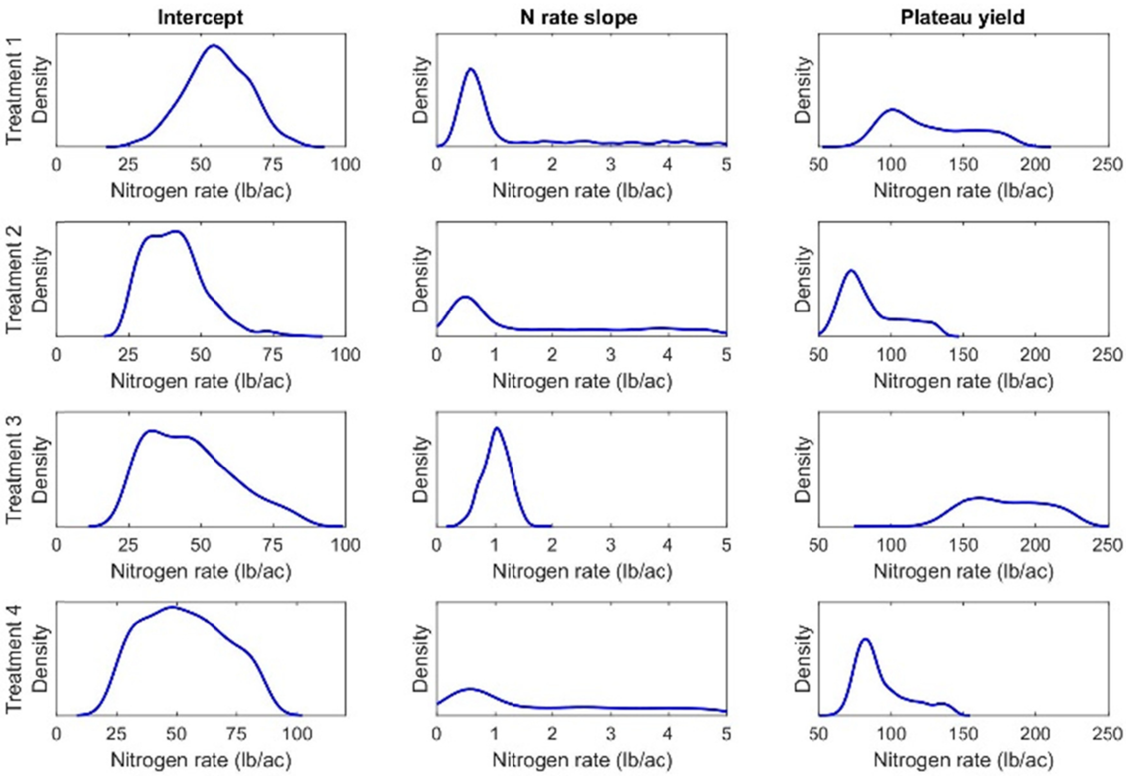


Figure 3. Posterior distributions of intercept, slope, and plateau parameters for model 3

When the per unit cost of nitrogen exceeds the marginal value product of an additional unit of nitrogen applied, then the decision rule is not to apply fertilizer. In the "apply 0 nitrogen" case, expected profit is the intercept term times the maize price, less per acre seed costs. Pairwise comparison of the intercept distributions suggests they are statistically different at the 1% level of significance (KS test results, Table 5). For treatment 1 (narrow plant; narrow row spacing), the expected maize yield is 55 bu ac$^{-1}$ with a profit of \$186 ac$^{-1}$ (Table 4). Expected profits for treatments 4, 3, and 2 were \$164, \$114, and \$139 ac$^{-1}$, respectively. Treatment 1 (narrow plant; narrow row spacing) dominated 2 by the first order. Treatment 4 (wide plant/wide row spacing) also dominated treatments 2 and 3 by the first order. The net return distributions of treatment 1 crossed the distributions of 3 and 4. In both cases, net returns for treatment 1 were higher at all levels of probability up to where the empirical distributions crossed (Figure 4). The cumulative area below the crossing points of 1 and 3, and 1 and 4 was greater than the area above the crossovers. Therefore, treatment 1 stochastically dominated treatments 3 and 4 by the second order; *i.e.*, $1 \succ (2, 3, 4)$, where "$\succ$" implies a risk-preferred set. By the same criterion, treatment 4 dominated treatments 2 and 3 by the second order; $4 \succ (2, 3)$. In summary, when the decision rule is "do not apply nitrogen", the risk-preferred ranking of the treatments is $1 \succ 4 \succ (2, 3)$.

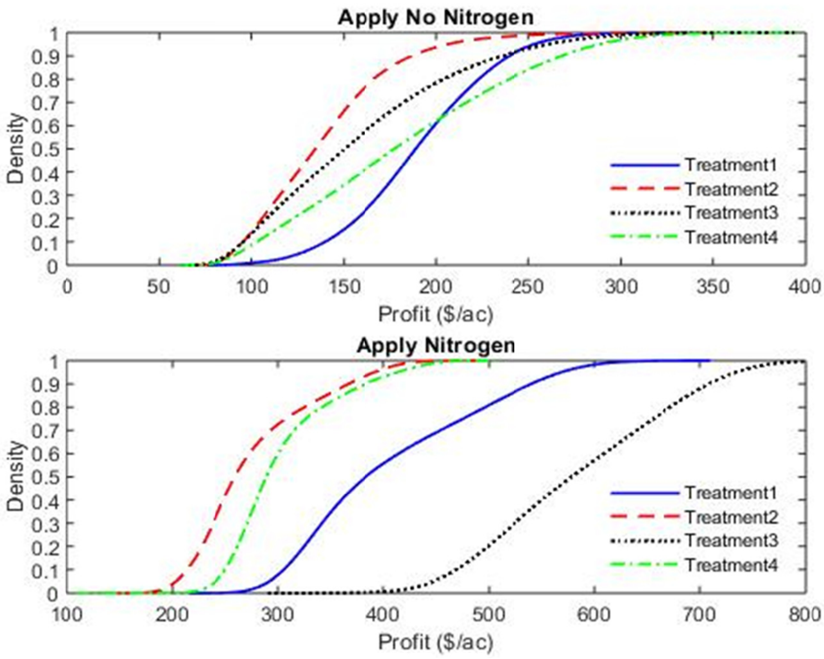


Figure 4. Empirical distributions of net returns

*Note*. Treatments indicate (plant; row) spacing: Treatment 1 is (0.49-ft; 1.67-ft), Treatment 2 is (0.49-ft; 2.49-ft), Treatment 3 is (0.98-ft; 1.67-ft), and Treatment 4 (0.98-ft; 2.49-ft).

Table 4. Maize Yield Response to Nitrogen under Different Plant and Row Spacing

| Estimate | Treatment 1 | Treatment 2 | Treatment 3 | Treatment 4 |
|---|---|---|---|---|
| Intercept | 55 | 41 | 33 | 48 |
| | [34; 76] | [25; 60] | [25; 76] | [25;82] |
| Linear response | 0.57 | 0.46 | 1.02 | 0.55 |
| | [0.24; 4.30] | [0.12; 4.58] | [0.58; 1.45] | [0.07; 4.66] |
| Plateau yield | 100 | 72 | 160 | 82 |
| | [87; 180] | [57; 128] | [139; 229] | [69; 136] |
| Error variance | 897 | 897 | 897 | 897 |
| | [690; 1,188] | [690; 1,188] | [690; 1,188] | [690; 1,188] |
| Optimal N rate (lbs ac$^{-1}$) | 80 | 68 | 125 | 62 |
| | [8; 244] | [4; 222] | [83; 198] | [0; 213] |
| Nitrogen applied: expected profit ($ ac$^{-1}$) | 338 | 243 | 528 | 282 |
| | [292; 514] | [200; 350] | [431; 684] | [237; 369] |
| No nitrogen applied: expected profit ($ ac$^{-1}$) | 186 | 139 | 114 | 164 |
| | [127; 249] | [91; 203] | [91; 255] | [94; 280] |

*Note*. Entries are the highest posterior densities (HPD, or modes) and HPD intervals of the distributions (1,000 iterations) with 5$^{th}$ and 95$^{th}$ credible intervals in brackets.

Table 5. Two-Sample Kolmogorov-Smirnov Test Statistics

| | Treatment 1 | Treatment 2 | Treatment 3 |
|---|---|---|---|
| Treatment | ---------------------------------- Intercept ---------------------------------- | | |
| 1 | - | | |
| 2 | 0.56 | - | |
| 3 | 0.36 | 0.21 | - |
| 4 | 0.20 | 0.38 | 0.19 |
| Treatment | ---------------------------------- Slope ---------------------------------- | | |
| 1 | - | | |
| 2 | 0.18 | - | |
| 3 | 0.43 | 0.41 | - |
| 4 | 0.22 | 0.11 | 0.46 |
| Treatment | ---------------------------------- Plateau ---------------------------------- | | |
| 1 | - | | |
| 2 | 0.63 | - | |
| 3 | 0.64 | 0.98 | - |
| 4 | 0.55 | 0.31 | 0.94 |
| Treatment | --------------------- Net returns: Nitrogen Applied --------------------- | | |
| 1 | - | | |
| 2 | 0.68 | - | |
| 3 | 0.65 | 0.99 | - |
| 4 | 0.59 | 0.36 | 0.96 |
| Treatment | ------------------- Net returns: No nitrogen Applied -------------------- | | |
| 1 | - | | |
| 2 | 0.56 | - | |
| 3 | 0.36 | 0.21 | - |
| 4 | 0.20 | 0.38 | 0.19 |

*Note*. All test statistics are significant at the 1% level.

Maize yield response to the first pound of nitrogen fertilizer applied was highest for treatment 3 (1.02 bu lb$^{-1}$ of nitrogen fertilizer), followed by treatments 1 and 4 (0.57 and 0.55 bu lb$^{-1}$ nitrogen fertilizer, respectively). The relative precision of the linear response estimate was greatest for treatment 3, with a range of 0.87 between the lower and upper bounds of the HDPI credible interval (Figure 3). The prior for the slopes of all treatments was a uniform distribution with a lower (upper) bound of 0 (5), with the upper bounds of the slope estimates for treatments 1, 2, and 4 approaching the prior upper bound of 5. The influence of the prior upper bound is apparent by inspection of the HDP credible intervals of treatments 1, 2, and 4. The variability of the posterior slope estimates for these treatments ranged between 4.06 (for treatment 1) to 4.59 (for treatment 4). Statistical comparison of the posterior distributions of the treatment slopes suggests the estimates are significantly different at the 1% level (Table 4).

The null hypothesis that the empirical distributions of the plateaus were similar was rejected at the 1% level of significance (Table 5). Treatment 3 (wide plant; narrow row spacing) produced the highest yield plateau (160 bu ac$^{-1}$) and the narrowest HDP credible region (range, 90) (Figure 3, Table 4). Treatment 3 > treatment 1 > treatment 4 > treatment 2 in terms of maize yield. This ranking suggests that narrow row spacing confers a yield advantage over wide row spacing under favorable maize and nitrogen prices.

If the marginal value product of an additional unit of nitrogen applied exceeds the cost per unit of nitrogen, the yield advantage gained from the narrow row spacing translates into the same ranking in terms of profitability. When the marginal physical product is greater than the ratio of per unit nitrogen costs and the maize price, the expected net return for treatment 3 is \$528 ac$^{-1}$. The optimal nitrogen rate corresponding with treatment 3's plateau was 125 lbs ac$^{-1}$. Visual inspection of the net returns' empirical distributions clearly indicates the following rank, in terms of profitability: treatment 3 > treatment 1 > treatment 4 > treatment 2 (Figure 4). Since none of the empirical distributions cross, we can conclude that the technology ranking for the risk-neutral and risk-averse individual would be the same. In other words, treatment 3 stochastically dominates treatments 1, 4, and 2 by the first degree. Likewise, treatment 1 dominates 4 and 2 by the first degree criteria, and treatment 4 FDSD-dominates 2.

## 4. Conclusion

The proof-of-concept exercise used data from a plot trial comparing maize response to nitrogen under different plant/row spacing. The treatments were unbalanced. Two of the plant/row spacing treatments received three levels of nitrogen, with one of those levels a check plot of 0 applied nitrogen. The remaining two plant/row spacing treatments did not receive a nitrogen control. Zero-rate check plots are important for identifying yield intercepts. In the case of the LRP model, yield intercepts are key components for identifying linear response/plateau join points. Yet, in the absence of check plots, intercepts are exceedingly difficult (or impossible) to determine, rendering the LRP under-identified. We approached this problem of limited information by using a Bayesian simulation procedure, infilling missing information with a relatively strong set of priors and leveraging information from the univariate statistics of the treatment yields. The statistical methods used here may be useful for informing larger plot or field experiments based on pilot trials with unbalanced, limited replications.

This paper also applied an alternative formulation of the linear response with plateau (LRP) model for estimating yield response to inputs. The LRP model "min" operator was reformulated as a latent threshold regression whereby yield-input pairs are sorted into response or plateau regimes. The reformulation reproduces exactly the same estimates one would observe in a LRP response model specified with the "min" operator. However, the linearization of the "min" operator allows for additional flexibility with the inclusion of qualitative variables, such as those that might be used to design experimental designs, soil types, management zones, or other discrete categories. In addition, linearization of the LRP's "min" problem provides additional insight into the economics of the input decision-making process under risk.

An advantage of the Bayesian estimation procedure used here is that the posterior distributions of the LRP parameters can be directly used to generate *ex-post* distributions for net returns and optimal nitrogen rates observed under different conditions or treatments. Using these distributions, and admitting some mild assumptions about human behavior and risk, the empirical distributions can be compared using non-parametric risk analysis procedures such as stochastic dominance.

Extension of this research could expand the linearized version of the LRP to accommodate nonlinear response functions and multiple inputs. Additional elaborations are conceivable, given the flexibility of linearization procedure, including modifications of the model for other knot-and-spline applications, or addressing discontinuous problems formulated as "max" problems. What remain unknown are computational efficiencies on gains by linearizing the family of LRP problems.

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## Notes

Note 1. Our overview closely follows Paris and Knapp (1989), Dillon and Anderson (1990), and Paris (1992).

Note 2. Tembo et al.'s (2008), Tumusiime et al.'s (2011), and Boyer et al.'s (2013) inclusion of yearly random effects in their LRP specification hybridizes cases 1 and 2.

Note 3. SAS PROC NLMIXED was used to estimate a LRP with maximum likelihood, while PROC MCMC was used to estimate the function using Bayesian MCMC.

Note 4. For example, PROC MCMC in SAS 9.4 (SAS Institute, Cary North Carolina), or *bayesmh* in STATA 15.0 (College Station, TX).

Note 5. Stillwater, Oklahoma reported an average temperature of 73 °F and average rainfall of 4.33 inches from April to September in 2017 and 2018 when the experiments were conducted.

Note 6. In principle, the KS test is a non-parametric *t*-test.

Note 7. Stochastic Efficiency with Respect to a Function (SERF) is another method to rank risky technology preferences if one is willing to assume a functional form that measures utility. Stochastic dominance is a nonparametric approach that does not require one to posit a utility function or risk aversion levels.

Note 8. We attempted to estimate the LRP using maximum likelihood in SAS's PROC NLMIXED, but the model did no converge and the standard errors could not be calculated. This occurred because of the limited degrees of freedom available in the trials we analysed.