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Population Dynamics in Optimally Controlled Economic Growth Models: Case of Cobb-Douglas Production Function

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Authors' contributions

This work was carried out in collaboration between all authors. Author SOB designed the study, performed the statistical analysis, wrote the protocol and the first draft of the manuscript, as well as managed the analyses of the study and the literature searches. Authors FTO and GAO supervised and provided the direction of analyses. All authors read and approved the final manuscript.

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Abstract

In this paper, we discuss optimally controlled economic growth models with Cobb-Douglas aggregate production function, comparing real per capita income performance in scenarios where the labour (population) growth dynamics range from purely exponential to strongly logistic. The paper seeks to ascertain, by means of analytical and qualitative methods, as well as numerical simulations, the causal factors and parameters, especially population related ones, which induce qualitative changes in the performance of real per capita income. The models use consumption per effective labour as their control variable, and capital per effective labour as the state variable. Income per effective labour is here used as the output variable. A time-discounted welfare functional is used as the objective functional, maximized subject to a differential equation in the state and control variables. Each system is found to be stable in the neighbourhood of its non-trivial critical value. The models are both locally controllable and observable. The models' simulation values, in control, state and output, appear plausible and consistent with reality. It is found out that under R & D technological process, economies with exponential population growth consistently out-perform those with logistic population growth. On the contrary, in all other instances, economies with exponential population growth consistently perform worse than those with logistic population growth. These findings have far reaching inferences with regard to the running of economies.

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1 Introduction

It is generally accepted in economic literature, for instance [1-7], that the time performance of gross domestic product (GDP), and thus, real per capita income of an economy, is based on the level and quality of labour, technology, physical and human capital. Even from an over simplified model incorporating a linear aggregate production function studied in [8], it is observed, among other things, that capital's share, as well as technology, in the aggregate production function is an essential factor in determining the performance of real per capita income. However, it is also realized in [8] that Malthus' concerns[1] in [9], which is at variance with [10][2], cannot be discounted, and so is that of [11][3].

The population growth dynamics, especially as the key parameter (carrying capacity) seems to govern, in most cases, how other system parameters impact on the performance of real per capita income over time in [8]. Invariably, economies with logistic population growth, as per the models, tend to experience higher real per capita income over time, in contrast with those with exponential population growth. The introduction of technology in the discussions in [8] appears to boost the performance of real per capita income, albeit marginally. Hence, Boserup's idea [11] of an economy inventing itself out of its difficulties through the use of technology also seems to be supported here. Are all these observations made in models wherein the aggregate production function is linear generally true in economics, where production functions are rather usually nonlinear? Consequently, there is the need to extend and formulate the models to include more realistic nonlinear production functions so as to address any associated problems arising from oversimplification.

Subsequently, this paper discusses the effects of population dynamics in optimally controlled economic growth models, using a generalised Cobb-Douglas aggregate production function of technology, capital, and labour. The local stability and controllability of the model(s) built is considered. It also brings to bear the concept of maximum sustainable population growth, its impact on economic performance. Furthermore, it implements qualitative analyses on the models with respect to their dependence on the system parameters, especially the population related ones. It also performs numerical simulations on the models to authenticate the theoretical results.

2 Theoretical Preliminaries

2.1 Optimal control theory and the Hamilton-Pontryagin equations

Let $x \in \mathcal{X} \subseteq R_+$, $u \in \mathcal{U} \subseteq R_+$ and $y \in \mathcal{Y} \subseteq R_+$ respectively denote the state, control, and output variables of a system of a continuous time-varying controlled system [12-16], such that

$$\dot{x}(t) = g(x(t), u(t), p, t) \tag{2.1.1}$$

and
$$y(t) = G(x(t), u(t), p, t). \tag{2.1.2}$$

where p denotes the set of system parameters. Let the objective (welfare) functional, W say, which is a function of the control $u(t)$ and state $x(t)$, just as in [8], be defined by

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[1] *Malthus (1798) states that, high population growth puts a lot of strain on economic performance, the presence of technology notwithstanding.*

[2] *Optimists such as George (1879) believe that through technological innovations and advancement, the earth's capability of containing humans is boundless.*

[3] *In contrast to these extreme views, Boserup (1965) states that population growth boosts technological growth, which in turn enhances economic well-being. Subsequently, as resources begin to run out, an economy is forced to invent its way out of the problem.*

$$W(x,u) = \int_{t_0}^{T_f} L(x(t),u(t),t)dt, \quad \text{for } T_f \geq 0. \tag{2.1.3}$$

Then the control problem reduces to finding the control $u(t)$ [8, 12, 13, 14, 16, 17, 18] that maximizes

$$W(x,u) = \int_{t_0}^{T_f} L(x(t),u(t),t)dt \quad \text{for } T_f \geq 0$$

subject to $\quad \dot{x}(t) = g(x(t),u(t),p,t), \quad$ for $x(t_0) = x_0 \geq 0$ and $x(T_f) = x_{T_f} \geq 0$,

for $\quad y(t) = G(x(t),u(t),t)$.

Hence, the related Hamiltonian function, H, is thus

$$H = H(x(t),u(t),\lambda(t),t) = L(x(t),u(t),t) + \lambda(t)g(x(t),u(t),p,t), \tag{2.1.4}$$

λ is the co-state function. The associated Hamilton-Pontryagin equations [19] are

$$H_x = L_x + \lambda g_x = -\dot{\lambda} \tag{2.1.5}$$

$$H_\lambda = g = \dot{x} \tag{2.1.6}$$

$$H_u = L_u + \lambda g_u = 0 \tag{2.1.7}$$

for $x(t_0) = x_0 \geq 0$, $x(T_f) = x_{T_f} \geq 0$ or $\lambda(T_f) = Px(T_f)$, for some $P \geq 0$.

2.2 Linear control problem

Assume $g(x(t),u(t),p,t)$ and $G(x(t),u(t),t)$ are linear or linearized, with system representation $\{a(t),b(t),c(t),d(t)\}$. Then the state and output equations [14,20] are respectively

$$\dot{x}(t) = a(t)x(t) + b(t)u(t) \tag{2.2.1}$$

$$y(t) = c(t)x(t) + d(t)u(t). \tag{2.2.2}$$

2.3 Stability, controllability and observability conditions

The scalar system in Equation (2.2.1) is stable if and only if $a(t) < 0$, for all t within the defined time interval [14,21,22]. [14], for instance, also suggests that the system is controllable within a given time interval if for any $x(t)$ there exists a control $u(t)$. Being controllable suggests the system is stabilizable, even if it not completely stable [14,20]. Equally, from [14], the above system is observable if for any output $y(t)$ there exists a state $x(t)$. The system is also detectable if it is observable. Additionally, there exit unique system solutions if the systems are stabilizable and detectable.

3 Model Development

3.1 Population (labour) growth dynamics

If a population (labour), $L(t)$, grows naturally at $0 < n < 1$, with a carrying capacity $\frac{1}{\sigma} > 0$. Then

$$\frac{d}{dt}L(t) = n\big(1 - \sigma L(t)\big)L(t) = N(L(t); n, \sigma). \tag{3.1.1}$$

Thus, the equilibria values are, $L_1 = 0$ and $L_2 = \frac{1}{\sigma} > 0$. Thus, $N'(L_1) = n > 0$, and hence, L_1 is an unstable equilibrium value. But $N'(L_2) = -n < 0$, implies L_2 is a stable equilibrium. For any initial population L_0 such that $L_1 < L_0 < L_2$, $L(t) > 0$ and $L(t) \to L_2$ as $t \to \infty$, and hence, $0 < L(t) < \frac{1}{\sigma}$, meaning $L(t)$ is bounded for all $t \geq 0$, given that $0 < n < 1$. Moreover, for any $L_0 > L_2$, $L(t)$ declines gradually to L_2 over time. Equation (3.1.1) suggests that when $\sigma = 0$, the dynamics of $L(t)$ tends exponential, as noted in [8]. Thus $L(t)$ bifurcates when $\sigma = 0$. $L(t)$ bifurcates again when $\sigma = 1$, as the trajectory tends constant over time when $\sigma = 1$, but declines to zero over time when $\sigma > 1$, for all $t \geq 0$. Variously stated, L_1 and L_2 are respectively a source[4], and a sink.

Assuming our starting time $t_0 = 0$, where the initial value L_0 is standardized to unity [8], then

$$L(t) = \frac{L_0 e^{nt}}{1+\sigma L_0(e^{nt}-1)} = \frac{e^{nt}}{1+\sigma(e^{nt}-1)} = \frac{e^{nt}}{(1-\sigma)+\sigma e^{nt}} \leq e^{nt} \tag{3.1.2}$$

$$\Rightarrow \qquad \frac{L'(t)}{L(t)} = \frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)} = \frac{n}{1+\frac{\sigma}{1-\sigma}e^{nt}} \leq n \tag{3.1.3}$$

since $\sigma \geq 0$, $0 < n < 1$ and for all $t \geq 0$, $1 - \sigma + \sigma e^{nt} = 1 + \sigma\left(nt + \frac{n^2t^2}{2} + \cdots\right) \geq 1$, and $e^{nt} \geq 1$.

<u>3.1.1 Sensitivity analysis on population dynamics</u>

From Equation (3.1.2), and as seen in [8], we obtain

$$\frac{\partial L}{\partial \sigma} = -\frac{(e^{nt}-1)e^{nt}}{[1+\sigma(e^{nt}-1)]^2} < 0 \tag{3.1.4}$$

$$\frac{\partial L}{\partial n} = \frac{(1-\sigma)te^{nt}}{[1+\sigma(e^{nt}-1)]^2} \begin{cases} > 0 & \text{for } 0 \leq \sigma < 1 \\ = 0 & \text{for } \sigma = 1 \\ < 0 & \text{for } \sigma > 1 \end{cases} \tag{3.1.5}$$

$$\text{and} \qquad \frac{\partial L}{\partial t} = \frac{(1-\sigma)ne^{nt}}{[1+\sigma(e^{nt}-1)]^2} \begin{cases} > 0 & \text{for } 0 \leq \sigma < 1 \\ = 0 & \text{for } \sigma = 1\,. \\ < 0 & \text{for } \sigma > 1 \end{cases} \tag{3.1.6}$$

From (3.1.4), $L(t)$ is a decreasing function of the parameter σ. Thus as $L(t)$ tends logistic, the lower its growth, and hence, its numbers in relation to the one which grows exponentially, all things being equal. From (3.1.5) and (3.1.6), $L(t)$ is an increasing function of n, and of time, t, when $0 \leq \sigma < 1$. But $L(t)$ is a decreasing function of n and t when $\sigma > 1$, and in the extreme, $L(t)$ decays down to zero. The greater the value of σ, the faster this is. However, $L(t)$ is a constant function of n and t when $\sigma = 1$. The referenced expressions, as well as (3.1.3), suggest that, for all $t > 0$, $L(t)$ grows the fastest when $\sigma = 0$, that is, when the growth dynamics is exponential. Like in [8], Fig. 3.1 and Fig. 3.2 illustrate this.

Generally, underdeveloped (lower income) economies and most developing (lower middle income) economies seem to experience exponential population growth, whereas the population growth in developed (high income) economies as well as most upper middle income economies is logistic. This can also be inferred from [8].

3.2 Technological growth dynamics

Technology $A(t)$ usually enhances labour to make it even more productive. Hence, assuming a generalized Cobb-Douglas aggregate production function, a type with physical capital and technology augmented labour, that guarantees balanced growth, for simplicity, then we have

$$Y(t) = F(K(t), A(t)L(t)) = \rho K^\alpha(t)[A(t)L(t)]^{1-\alpha} \tag{3.2.1}$$

[4] *Additionally,* $N'(L_1) = n < 0$, *for all* $n < 0$. *Hence,* $n = 0$ *is a bifurcation value, and in contrast with the earlier discussion,* $L(t) \to 0$ *as* $t \to \infty$ *when* $n < 0$, *for* $L_0 > 0$.

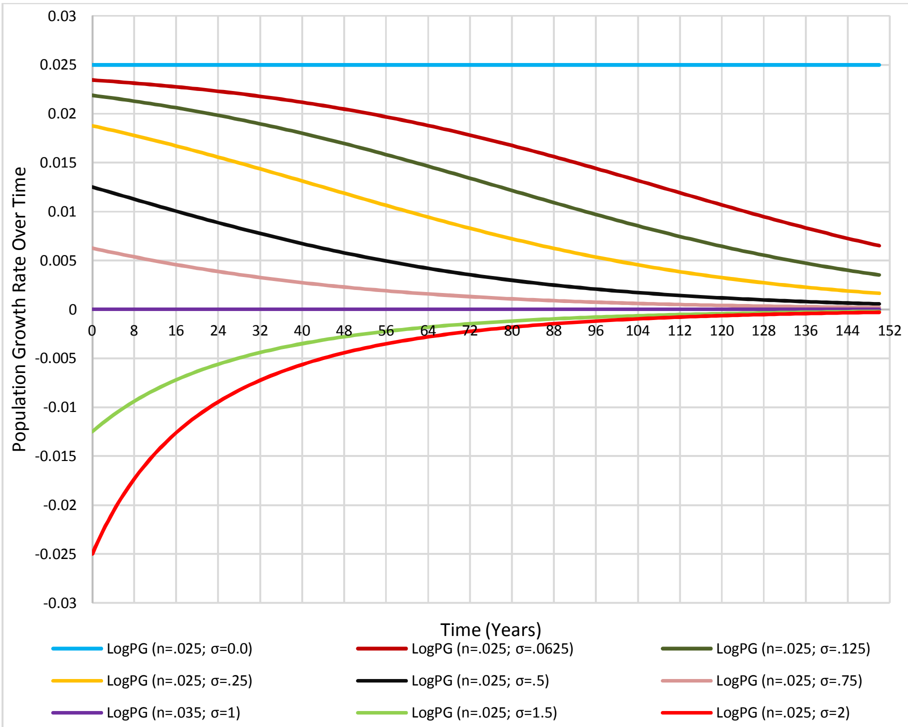

Fig. 3.1. Population growth dynamics over time for varying σ values

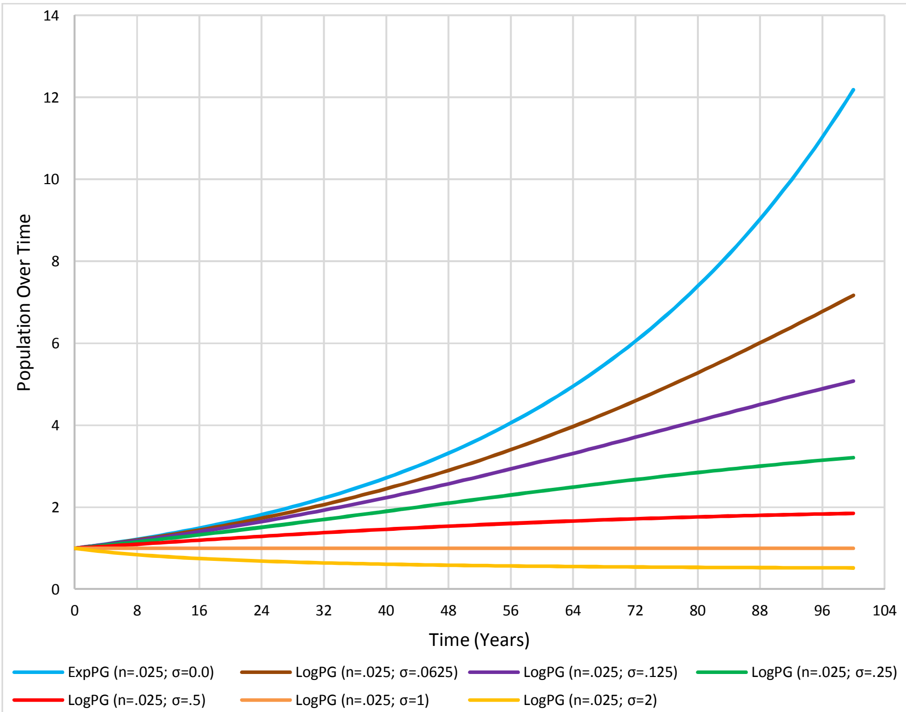

Fig. 3.2. Population dynamics over time for varying σ values

$$\Rightarrow \qquad \frac{\dot{Y}(t)}{Y(t)} = \alpha\frac{\dot{K}(t)}{K(t)} + (1-\alpha)\left[\frac{\dot{A}(t)}{A(t)} + \frac{\dot{L}(t)}{L(t)}\right] \tag{3.2.2}$$

$$\text{i.e.,} \qquad \frac{\dot{A}(t)}{A(t)} = \frac{1}{1-\alpha}\left[\delta - \alpha\delta_K - (1-\alpha)\frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}\right] \tag{3.2.3}$$

for $\delta = \dot{Y}/Y$, $\delta_K = \dot{K}/K$, and $\frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)} = \dot{L}/L$. In a balanced growth, $\delta = \delta_K$ [23,24]. Hence, the modified residual technological progress, $A(t)$, is defined by

$$\frac{\dot{A}(t)}{A(t)} = \delta - \frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}. \tag{3.2.4}$$

But from [25] and [26], $L(t)$ may be segregated into two additive parts, that is, $L(t) = L_Y(t) + L_A(t)$, where L_A (L_Y) is the labour used in the research (actual production) sector to create new technology or advance existing ones. If $0 < \phi \leq 1$ is the research sector's average productivity, and $0 < \theta < 1$, the fraction of existing technology used to produce new one(s), then for $\dot{L}_A/L_A = \dot{L}/L = \frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}$, the modified research and development (modified R & D) technological process, $A(t)$, is given by

$$A(t) = \phi A^{\theta}(t)L_A(t) \qquad \Rightarrow \qquad A^{1-\theta}(t) = \phi L_A(t) \tag{3.2.5}$$

$$\Rightarrow \qquad \frac{\dot{A}(t)}{A(t)} = \frac{\phi n(1-\sigma)}{(1-\theta)[1+\sigma(e^{nt}-1)]}. \tag{3.2.6}$$

Let A_0 be the initial value of A. Then by respectively putting A_1, and A_2 for A in Equations (3.2.4) and (3.2.6), each with initial value A_0 normalised to unity, we obtain

$$A_1(t) = A_0[1+\sigma(e^{nt}-1)]e^{(\delta-n)t} = [1+\sigma(e^{nt}-1)]e^{(\delta-n)t} \tag{3.2.7}$$

$$\text{and} \qquad A_2(t) = A_0\left[\frac{e^{nt}}{1+\sigma(e^{nt}-1)}\right]^{\frac{\phi}{1-\theta}} = \left[\frac{1}{(1-\sigma)e^{-nt}+\sigma}\right]^{\frac{\phi}{1-\theta}}. \tag{3.2.8}$$

Assuming a technological growth dynamics similar to that of L such that it has a natural growth $0 < a < 1$ with a carrying capacity $\frac{1}{\xi} > 0$. Then the logistic form of technology growth, $A_3(t)$, is

$$\frac{d}{dt}A_3(t) = a(1-\xi A_3(t))A_3(t). \tag{3.2.9}$$

It has similar qualitative properties like $L_2(t)$, spurs higher growth in real per capita GDP when ξ is very small, greatest when $\xi = 0$ (in which growth rate is kept constant at a), and yields

$$A_3(t) = \frac{A_0e^{at}}{1+A_0\xi(e^{at}-1)} = \frac{1}{(1-\xi)e^{-at}+\xi}, \qquad \text{for } A_0 = 1. \tag{3.2.10}$$

3.2.1 Sensitivity analysis on technological growth dynamics

From Equation (3.2.7), and for all $t > 0$, we obtain

$$\frac{\partial A_1}{\partial t} = [\delta - n(1-\sigma) + (e^{nt}-1)\delta\sigma]e^{(\delta-n)t} > 0 \tag{3.2.11}$$

$$\frac{\partial A_1}{\partial \sigma} = (e^{nt}-1)e^{(\delta-n)t} > 0 \tag{3.2.12}$$

and $$\frac{\partial A_1}{\partial n} = -(1-\sigma)te^{(\delta-n)t} \begin{cases} < 0 & \text{for } 0 \le \sigma < 1 \\ = 0 & \text{for } \sigma = 1 \\ > 0 & \text{for } \sigma > 1 \end{cases}. \tag{3.2.13}$$

Similarly, for all $t > 0$, and the given domain of θ and ϕ, Equation (3.2.8) also gives

$$\frac{\partial A_2}{\partial \phi} \cong \frac{(1-\sigma)nt}{1-\theta} A_2(t) \begin{cases} > 0 & \text{for } 0 \le \sigma < 1 \\ = 0 & \text{for } \sigma = 1 \\ < 0 & \text{for } \sigma > 1 \end{cases} \tag{3.2.14}$$

$$\frac{\partial A_2}{\partial \theta} \cong \frac{(1-\sigma)n\phi}{(1-\theta)^2} A_2(t) \begin{cases} > 0 & \text{for } 0 \le \sigma < 1 \\ = 0 & \text{for } \sigma = 1 \\ < 0 & \text{for } \sigma > 1 \end{cases} \tag{3.2.15}$$

$$\frac{\partial A_2}{\partial n} = \frac{1-\sigma}{1-\theta} \cdot \frac{\phi t}{1+\sigma(e^{nt}-1)} A_2(t) \begin{cases} > 0 & \text{for } 0 \le \sigma < 1 \\ = 0 & \text{for } \sigma = 1 \\ < 0 & \text{for } \sigma > 1 \end{cases} \tag{3.2.16}$$

$$\frac{\partial A_2}{\partial t} = \frac{1-\sigma}{1-\theta} \cdot \frac{\phi n}{1+\sigma(e^{nt}-1)} A_2(t) \begin{cases} > 0 & \text{for } 0 \le \sigma < 1 \\ = 0 & \text{for } \sigma = 1 \\ < 0 & \text{for } \sigma > 1 \end{cases} \tag{3.2.17}$$

and $$\frac{\partial A_2}{\partial \sigma} = -\frac{\phi}{1-\theta} \cdot \frac{e^{nt}-1}{1+\sigma(e^{nt}-1)} A_2(t) < 0. \tag{3.2.18}$$

On the other hand, for all $t > 0$, Equation (3.2.10) gives

$$\frac{\partial A_3}{\partial t} = \frac{(1-\xi)a}{1+\xi(e^{at}-1)} A_3(t) \begin{cases} > 0 & \text{for } 0 \le \xi < 1 \\ = 0 & \text{for } \xi = 1 \\ < 0 & \text{for } \xi > 1 \end{cases} \tag{3.2.19}$$

$$\frac{\partial A_3}{\partial a} = \frac{(1-\xi)a}{[1+\xi(e^{at}-1)]^2} e^{at} \begin{cases} > 0 & \text{for } 0 \le \xi < 1 \\ = 0 & \text{for } \xi = 1 \\ < 0 & \text{for } \xi > 1 \end{cases} \tag{3.2.20}$$

and $$\frac{\partial A_3}{\partial \xi} = -\frac{(e^{at}-1)}{[1+\xi(e^{at}-1)]^2} e^{at} < 0. \tag{3.2.21}$$

Expressions (3.2.11) and (3.2.12) suggest that, for all $t > 0$, the residual technological process, $A_1(t)$, is an increasing function of time, t, as well as the parameter σ. Subsequently, over time, the higher the value of σ, the higher the growth prospects ignited by the technological process $A_1(t)$ in real per capita income, all things being equal. Expression (3.2.13) indicates that $A_1(t)$ is an increasing function of the parameter n only when $\sigma > 1$, a condition that has dire consequences for the very sustenance of population. On the other hand, $A_1(t)$ is a decreasing function of the parameter n when $0 \le \sigma < 1$, but a constant function of n when $\sigma = 1$. It is also quite clear from Expressions (3.2.4) and (3.2.7) that $A_1(t)$ is also an increasing function of δ, the growth rate of GDP. Moreover, anytime there is a positive difference between δ and n, that is, $\delta - n > 0$, $A_1(t)$ is likely to ignite growth in real per capita GDP.

With regards to the R & D technological process, $A_2(t)$, Expressions (3.2.14) to (3.2.17) respectively suggest that, for all $t > 0$, $A_2(t)$ is an increasing function of the parameters ϕ, θ, n (unlike in $A_1(t)$ discussed earlier), and time t, whenever $0 \le \sigma < 1$. However, $A_2(t)$ becomes a decreasing function of ϕ, θ, n and t, when $\sigma > 1$, whereas it is a constant function of ϕ, θ, n and t, whenever $\sigma = 1$, for all $t > 0$. Expression (3.2.18), on the other hand, suggests that, for all $t > 0$, $A_2(t)$ is a decreasing function of σ. Hence, as population growth becomes more and more logistic, the R & D technological process becomes inimical to real per capita income growth. Consequently, or by inference from Expressions (3.2.14) to

(3.2.18), the R & D technological process possibly incites the greatest growth prospects in real per capita income when population growth is exponential, that is, when $\sigma = 0$.

Expressions (3.2.19) and (3.2.20) indicate that, for all $t > 0$, the logistic formulation of technology, $A_3(t)$, is an increasing function of the parameter a, and time t, whenever $0 \leq \xi < 1$. However, $A_3(t)$ is decreasing function of a and t, when $\xi > 1$. On the other hand, $A_3(t)$ is a constant function of a and t, whenever $\xi = 1$. But $A_3(t)$ is decreasing function of ξ, for all $t > 0$, as in (3.2.21). Thus $A_3(t)$ is most likely to generate the greatest growth in real per capita whenever $\xi = 0$, as alluded to earlier.

3.3 Optimal growth model in a closed economy

In an economy in which income, $Y(t)$, is either expended on consumption, $C(t)$, or investment, $I(t)$ [8, 17, 23, 27-32], we have

$$Y(t) = C(t) + I(t). \tag{3.3.1}$$

Assume the aggregate production function

$$Y(t) = Y\big(K(t), A(t)L(t)\big). \tag{3.3.2}$$

From theory, if μ is the rate of depreciation of $K(t)$, then in simplest form [18, 33], $I(t)$ is given by

$$I(t) = \dot{K}(t) + \mu K(t). \tag{3.3.3}$$

Now, putting $\hat{y} = Y/AL$, $\hat{c} = C/AL$, and $\hat{k} = K/AL$, assuming technology grows at a constant rate $0 < a < 1$, and using the transformations as seen in [8], for simplicity, then

$$\dot{\hat{k}}(t) = g(\hat{k}(t), \hat{c}(t), t) = f\big(\hat{k}(t)\big) - \left(a + \mu + \tfrac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}\right)\hat{k}(t) - \hat{c}(t) \tag{3.3.4}$$

$$\text{and} \qquad \hat{y}(t) = f\big(\hat{k}(t)\big). \tag{3.3.5}$$

Equations (3.3.4), and (3.3.5) are respectively the state and output equations, $\hat{k}(t)$ is the state variable, $\hat{c}(t)$ is the control variable [8, 13, 18, 27, 34, 35, 36]. Using Equation (3.2.1), and the transformations in [8], where $0 < \alpha < 1$ share of K in Y, and $\rho \geq 1$ the unaccounted for factors in Y, then

$$\dot{\hat{k}}(t) = g(\hat{k}(t), \hat{c}(t), t) = \rho\hat{k}^{\alpha}(t) - \left(a + \mu + \tfrac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}\right)\hat{k}(t) - \hat{c}(t). \tag{3.3.6}$$

Alternatively, if we use the idea that income is either consumed or saved, or that investment is identically equal to savings, $S(t)$, that is, $I(t) = S(t)$, and that savings is a fraction of income, then

$$S(t) = sY(t) \tag{3.3.7}$$

$$\text{i.e.,} \qquad C(t) = (1-s)Y(t) \qquad \Rightarrow \qquad \hat{c}(t) = (1-s)\hat{y}(t), \tag{3.3.8}$$

where s is the savings rate, or the propensity to save [23]. Thus Equations (3.3.6) and (3.3.5) become

$$\dot{\hat{k}}(t) = \rho s\hat{k}^{\alpha}(t) - \left(a + \mu + \tfrac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}\right)\hat{k}(t) \tag{3.3.9}$$

and $$\hat{y}(t) = \rho s\hat{k}^{\alpha}(t). \tag{3.3.10}$$

3.3.1 The utility and welfare functional

The utility per head, $\upsilon(t)$, and welfare functional, $W(\hat{c})$, are respectively given by

$$\upsilon(t) = u(\hat{c}(t)), \tag{3.3.11}$$

$$W(\hat{c}) = \int_{t_0}^{T_f} \frac{e^{-(\gamma-n)\tau} u(\hat{c}(\tau))}{1+\sigma(e^{n\tau}-1)} d\tau, \qquad T_f \geq t_0. \tag{3.3.12}$$

for $\gamma - n > 0$, $u'(\hat{c}(t)) > 0$, $u''(\hat{c}(t)) < 0$ [23, 33, 34, 35] and $0 < \gamma < 1$ is discount rate of value.

3.4 Equilibrium and linearization analyses

At the equilibrium, $\dot{\hat{k}}(t) = 0$, Equation (3.3.9), gives the equilibria values of $\hat{k}(t)$ thus

$$\hat{k}_{e_1}^*(t) = 0 \qquad \text{or} \qquad \hat{k}_{e_2}^*(t) = \left(\frac{\rho s}{a+\mu+\frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}}\right)^{\frac{1}{1-\alpha}} \tag{3.4.1}$$

The equilibrium value $\hat{k}_{e_1}^*$ gives corresponding equilibria values of $\hat{c}(t)$ and $\hat{y}(t)$ as zero, which are of little or no interest. Using $\hat{k}_{e_2}^*$, the equilibria values of $\hat{y}(t)$ and $\hat{c}(t)$ respectively becomes

$$\hat{y}_e^*(t) = \rho\left(\frac{\rho s}{a+\mu+\frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}}\right)^{\frac{\alpha}{1-\alpha}} \quad \text{and} \quad \hat{c}_e^*(t) = \rho(1-s)\left(\frac{\rho s}{a+\mu+\frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}}\right)^{\frac{\alpha}{1-\alpha}}. \tag{3.4.2}$$

Equations (3.4.1) and (3.4.2) suggest that the higher the values of s, ρ and α, the greater the equilibrium value of $\hat{k}(t)$, and hence, that of $\hat{y}(t)$, and vice versa. Now, linearizing around the equilibrium, gives

$$\hat{y}(t) \approx \rho(1-\alpha)\left(\hat{k}_{e_2}^*\right)^{\alpha} + \rho\alpha\left(\hat{k}_{e_2}^*\right)^{\alpha-1}\hat{k}(t) \approx \psi(t) + \frac{\alpha}{s}\left[a+\mu+\frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}\right]\hat{k}(t) \tag{3.4.3}$$

$$\Rightarrow \qquad \hat{z}(t) \approx \frac{\alpha}{s}\left(a+\mu+\frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}\right)\hat{k}(t) = \varphi(t)\hat{k}(t) \tag{3.4.4}$$

$\hat{z}(t) = \hat{y}(t) - \psi(t)$, $\psi(t) = (1-\alpha)\hat{y}^*(t)$ and $\varphi(t) = \frac{\alpha}{s}\left(a+\mu+\frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}\right)$. For $\hat{c}_1(t) = \hat{c}(t) - \psi(t)$

$$\dot{\hat{k}}(t) \approx \left(\frac{\alpha}{s}-1\right)\left(a+\mu+\frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}\right)\hat{k}(t) - \hat{c}_1(t) = g(\hat{k}(t), \hat{c}_1(t), t). \tag{3.4.5}$$

3.5 Local controllability observability and stability of the linearized system

Given that for any state $\hat{k}(t)$ in Equation (3.4.5), there exists a control $\hat{c}_1(t)$, and hence, $\hat{c}(t)$, that drives the system to a desired state. Hence the linearized system is completely controllable, in the neighbourhood of the critical value [14]. But given that it is completely controllable, it is also stabilizable. Similarly, from Equation (3.4.4), the system is completely observable, and hence, detectable [14].
The system is asymptotically stable if and only if $\alpha < s$. Granted that $\alpha < s$ cannot be guaranteed always, our system might not generally be stable. But from Equation (3.3.10), and restated as

$$\dot{\hat{k}}(t) = \rho s \hat{k}^{\alpha}(t) - \left(a+\mu+\frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}\right)\hat{k}(t) = f_{\text{para}}(\hat{k}(t)) \tag{3.5.1}$$

gives the equilibria values $\hat{k}_0$ and $\hat{k}_{e_2}^*$, as provided in Equation (3.4.1). Similar to the discussions in Section 3.1, $\hat{k}(t) = \hat{k}_{e_1}^*$ is a source, and $\hat{k}(t) = \hat{k}_{e_2}^*$ is a sink, since for any initial value $\hat{k}(t_0)$ such that $\hat{k}_{e_1}^* <$

$\hat{k}(t_0) < \hat{k}^*_{e_2}$, which is in the domain of interest, $\hat{k}(t) > 0$ for all $t \geq 0$, and $\hat{k}(t) \to \hat{k}^*_{e_2}$ as $t \to \infty$. Thus for any $\hat{k}^*_{e_1} < \hat{k}(t_0) < \hat{k}^*_{e_2}$, $\hat{k}(t)$ is bounded, that is, $\hat{k}^*_{e_1} < \hat{k}(t) < \hat{k}^*_{e_2}$. For any initial value $\hat{k}(t_0) > \hat{k}^*_{e_2}$, $\hat{k}(t)$ decays down to $\hat{k}^*$. Hence, the application of higher and higher initial values will lead to less than proportionate increase in values of $\hat{k}(t)$. This assertion is underlain by the fact that

$$f'_{\text{para}}\left(\hat{k}^*_{e_2}\right) = -(1-\alpha)\left(a + \mu + \frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}\right) < 0 \tag{3.5.2}$$

since $1 - \alpha > 0$, and $a + \mu + \frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)} > 0$, for all $t \geq 0$. Hence, $\hat{k}$ is stable in the neighbourhood of the equilibrium value $\hat{k}^*_{e_2}$ [14, 21, 22]. Hence, $\hat{k}^*_{e_2}$ is a sink. But in the rare situation where $\sigma \gg 1$ such that $\sigma > \frac{a+n+\mu}{n-(a+\mu)(e^{nt}-1)}$, then $\hat{k}^*_{e_2}$ is an unstable equilibrium value.

3.6 Sensitivity and bifurcation analyses of the systems

From the above, the trajectory of $\hat{k}(t)$, and hence, $k(t)$ and $y(t)$, bifurcates when $a + \mu + \frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)} = 0$. Let the equilibrium trajectories, per labour, of capital and income be $k^*(t)$ and $y^*(t)$ respectively, then

$$k^*(t) = \hat{k}^*_{e_2}(t)A(t) = A_0\left(\frac{\rho s}{a+\mu+\frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}}\right)^{\frac{1}{1-\alpha}} e^{at} \tag{3.6.1}$$

$$\Rightarrow \qquad \dot{k}^*(t) = \left[a + \frac{1}{1-\alpha}\cdot\frac{n^2(1-\sigma)e^{nt}}{(a+\mu)[1+\sigma(e^{nt}-1)]+n(1-\sigma)}\cdot\frac{\sigma}{1+\sigma(e^{nt}-1)}\right]k^*(t) \begin{cases} > a, & 0 < \sigma < 1 \\ = a, & \text{for } \sigma = 0,1 \\ < a, & \text{for } \sigma > 1 \end{cases} \tag{3.6.2}$$

and

$$y^*(t) = \hat{y}^*_e(t)A(t) = \rho_0\left(\frac{\rho s}{a+\mu+\frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}}\right)^{\frac{\alpha}{1-\alpha}} e^{at} \tag{3.6.3}$$

$$\Rightarrow \qquad \dot{y}^*(t) = \left[a + \frac{\alpha}{1-\alpha}\cdot\frac{n^2(1-\sigma)e^{nt}}{(a+\mu)[1+\sigma(e^{nt}-1)]+n(1-\sigma)}\cdot\frac{\sigma}{1+\sigma(e^{nt}-1)}\right]y^*(t) \begin{cases} > a, & 0 < \sigma < 1 \\ = a, & \text{for } \sigma = 0,1. \\ < a, & \text{for } \sigma > 1 \end{cases} \tag{3.6.4}$$

$\rho_0 = A_0\rho$. The time trajectory of k, and hence, y, bifurcates whenever $n = 0$, or $a = 0$, or $\sigma = 0$, or $\sigma = 1$. Invariably, y remains constant whenever $\sigma = 0, 1$ and $a = 0$; decay down to zero when $a < 0$, for all $\sigma > 1$ or $\sigma = 0, 1$; and grows up when $a > 0$, for all $0 < \sigma < 1$. When $a = 0$, then the time trajectory of y may rise, probably marginally, when $0 < \sigma < 1$; but rises initially, then stay constant when $\sigma = 0$, or fall when $\sigma > 1$. Let $m = a + \frac{\alpha}{1-\alpha}\cdot\frac{n^2(1-\sigma)\sigma e^{nt}}{(a+\mu)[1+\sigma(e^{nt}-1)]^2+n(1-\sigma)[1+\sigma(e^{nt}-1)]}$, then for all $t \geq 0$,

$$\frac{\partial y^*}{\partial a} = \left[t - \frac{\alpha}{1-\alpha}\cdot\frac{1+\sigma(e^{nt}-1)}{(a+\mu)[1+\sigma(e^{nt}-1)]+(1-\sigma)n}\right]y^* > 0 \tag{3.6.5}$$

$$\frac{\partial m}{\partial \alpha} = \frac{\sigma}{(1-\alpha)^2}\cdot\frac{n^2(1-\sigma)e^{nt}}{(a+\mu)[1+\sigma(e^{nt}-1)]^2+n(1-\sigma)[1+\sigma(e^{nt}-1)]} \begin{cases} > 0, & \text{for } 0 < \sigma < 1 \\ = 0, & \text{for } \sigma = 0, 1. \\ < 0, & \text{for } \sigma > 1 \end{cases} \tag{3.6.6}$$

We also have

$$\frac{\partial y^*}{\partial \sigma} = \frac{\alpha}{1-\alpha}\cdot\frac{ne^{nt}}{(a+\mu)[1+\sigma(e^{nt}-1)]+n(1-\sigma)}\cdot\frac{y^*}{[1+\sigma(e^{nt}-1)]} > 0 \tag{3.6.7}$$

$$\frac{\partial y^*}{\partial s} = \frac{1}{s}\left(\frac{\alpha}{1-\alpha}\right) y^* > 0 \qquad \text{and} \qquad \frac{\partial y^*}{\partial \rho} = \frac{1}{\rho}\left(\frac{1}{1-\alpha}\right) y^* > 0 \tag{3.6.8}$$

$$\frac{\partial y^*}{\partial n} = \frac{\alpha}{1-\alpha} \cdot \frac{(1-\sigma)[\{1+(nt-1)e^{nt}\}\sigma-1]}{[1+\sigma(e^{nt}-1)][(a+\mu)[1+\sigma(e^{nt}-1)]+(1-\sigma)n]} y^* \begin{cases} > 0, & \text{for } 0 < \sigma < 1 \\ = 0, & \text{for } \sigma = 1 \\ < 0, & \text{for } \sigma = 0 \text{ or } \sigma > 1 \end{cases} \tag{3.6.9}$$

and
$$\frac{\partial y^*}{\partial \mu} = -\frac{\alpha}{1-\alpha} \cdot \frac{1+\sigma(e^{nt}-1)}{(a+\mu)[1+\sigma(e^{nt}-1)]+(1-\sigma)n} y^* < 0. \tag{3.6.10}$$

From (3.6.5) and Equation (3.6.4), we can conveniently conclude that the higher the value of a, the faster y grows, and hence, a has a positive effect on the time-values of y. The converse is also true. Equations (3.6.6) and (3.6.4) suggest that for all $0 < \sigma < 1$, higher values of α ignite faster growth in y^*, and hence, in y, and vice versa. However, whenever $\sigma = 0$ or $\sigma = 1$, the growth effect of α on y is kept at zero, and hence, y remains constant, except if there is a positive technical progress, as seen in Equation (3.6.4), in which case y grows over time. When $\sigma > 1$, then higher values of α is disincentive for growth in y. Under such instance, y may grow or the negative growth slowed down if there exists a positive technical growth rate a, as per Equation (3.6.4). For all that it is worth in terms of real per capita income time values, $\sigma > 1$ is a recipe for population extinction, and thus, undesirable.

From (3.6.8), s, and hence, ρ, has a direct positive effect on y, and the higher each one is, the higher the time-values of y. The division by s suggests that as s becomes higher and higher, the growth potential generated diminishes. From Equation (3.4.5), the time trajectory of $\hat{k}$, and by implication, y, may experience the fastest growth, before the attainment of equilibrium, whenever $\alpha > s$. The highest this difference is the greatest the growth potential. This growth, before equilibrium, is negated whenever $\alpha < s$, but this may be off-set by the developments after the equilibrium, as well as the tendency of establishing higher equilibrium value.

However, from (3.6.9), the economy starting from a reference point of a higher value of n will be instrumental for the establishment of higher real per capita income trajectory if the population growth dynamics turns logistic, and moreover $0 < \sigma < 1$. But higher values of n are inimical to the generation of higher real per capita income performance when $\sigma = 0$. Although $\frac{\partial y^*}{\partial n} < 0$ when $\sigma > 1$, given that higher values of σ ignite higher real per capita GDP time values, per (3.6.7), the negative effect of n herein may be eliminated or greatly reduced, except for the distasteful impart of $\sigma > 1$ on population as discussed earlier. But n has neutral effect on real per capita GDP growth when $\sigma = 1$. (On the other hand, higher values of μ, affect the time-values of y negatively, as per (3.6.10), most likely due to its effect of establishing lower equilibrium value of y. The converse is similarly true.)

Alternatively, from the above, we define per capita income, $y(t)$, thus

$$y(t) = A(t) f(\hat{k}(t)) \tag{3.6.11}$$

$$\Rightarrow \qquad \frac{\dot{y}(t)}{y(t)} = a + \varepsilon_{\hat{k}}(\hat{k}(t)) \frac{\dot{\hat{k}}(t)}{\hat{k}(t)} = a + \alpha \frac{\dot{\hat{k}}(t)}{\hat{k}(t)} \tag{3.6.12}$$

where $\alpha = \varepsilon_{\hat{k}}(\hat{k}(t)) = \frac{f'(\hat{k}(t))\hat{k}(t)}{f(\hat{k}(t))} \in (0,1)$ is the elasticity of the production function f, and measures the share of capital in the production mix. But from Equation (3.4.4), we have

$$\frac{\dot{\hat{k}}(t)}{\hat{k}(t)} = \frac{f(\hat{k}(t))}{\hat{k}(t)} - \left(a + \mu + \frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}\right) - \frac{\hat{c}(t)}{\hat{k}(t)} = \frac{sf(\hat{k}(t))}{\hat{k}(t)} - \left(a + \mu + \frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}\right). \tag{3.6.13}$$

Taking first order Taylor's expansion of Equation (3.6.13), around $\hat{k}^*_{e_2}$, with respect to $\ln(\hat{k}(t))$, then

$$\frac{\dot{\hat{k}}(t)}{\hat{k}(t)} \approx \left(\varepsilon_{\hat{k}}\left(\hat{k}_{e_2}^*\right)-1\right)\left(a+\mu+\frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}\right)\left(\ln(\hat{k}(t))-\ln(\hat{k}_{e_2}^*)\right) \tag{3.6.14}$$

since at the steady state $\frac{sf(\hat{k}_{e2}^*)}{\hat{k}_{e_2}^*}=\left(a+\mu+\frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}\right)$. Hence, Equation (3.6.14) becomes

$$\frac{\dot{y}(t)}{y(t)} \approx a-\varepsilon_{\hat{k}}(\hat{k}_{e_2}^*)\left(1-\varepsilon_{\hat{k}}(\hat{k}_{e_2}^*)\right)\left(a+\mu+\frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}\right)\left(\ln(\hat{k}(t))-\ln(\hat{k}_{e_2}^*)\right). \tag{3.6.15}$$

Now, from the steady state analysis, the steady state output or real GDP per capita is

$$y^*(t)=f\left(\hat{k}_{e_2}^*(t)\right)A(t)=\hat{y}_e^*(t)A(t). \tag{3.6.16}$$

A first-order Taylor expansion of $\ln(y^*(t))$ with respect to $\ln(\hat{k}(t))$ in the locality of $\ln(\hat{k}^*)$ gives

$$\ln(y(t))-\ln(y^*(t)) \approx \varepsilon_{\hat{k}}(\hat{k}_{e_2}^*)\left[\ln(\hat{k}(t))-\ln(\hat{k}_{e_2}^*)\right]. \tag{3.6.17}$$

Putting Equation (3.6.17) into that of (3.6.15) gives

$$\frac{\dot{y}(t)}{y(t)} \approx a-(1-\alpha)\left(a+\mu+\frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}\right)[\ln(y(t))-\ln(y^*(t))]. \tag{3.6.18}$$

From Equation (3.6.18), the per capita income grows much faster, greater than a, any time per capita income is less than the system established equilibrium per capita income, that is, $y(t)<y^*(t)$, since $1-\alpha>0$ and $a+\mu+\frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}>0$, for $t\geq 0$. It grows at a rate less than the technological progress rate, a, anytime $y(t)>y^*(t)$, and with or without technological progress, growth in $y(t)$ here may hit the negative (occasionally). The growth rate in $y(t)$ is just a whenever $y(t)=y^*(t)$, and zero when $a=0$. Thus in the absence of technological progress, or if it is negative, growth in $y(t)$ is much less than discussed above. If $a<0$, then per capita GDP values may tumble over time. The expression on the right side of Equation (3.6.18), excluding the first term, a, is termed the rate of convergence.

3.7 Analysis on maximum sustainable population growth

Just like noted in [8], the factor shares (excluding labour) in the production mix positively influence real per capita income performance, whereas depreciation rates and population growth rate do the very opposite. Thus the most prudent policy for exiting the Malthusian trap is to pursue measures that enhance technological growth, savings rates and marginal products per (effective) labour, lower depreciation rates and shares of natural resources in the production mix. In contrast to [37], here the MSPG is rather an increasing function of technological growth, marginal products of factors of production (excluding labour) and their corresponding rates of savings, but a decreasing function of the factor depreciation rates and associated factor shares.

3.8 The Hamilton-Pontryagin equations for the Linearized systems

From Equations (2.3.2) to (2.3.5), and using a logarithmic utility functional, $u(c(t))=\ln c(t)$, taking $t_0=0$, for simplicity, the associated Hamilton-Pontryagin equations give

$$\hat{c}(t)=\frac{e^{-(\gamma-n)t}}{[1+\sigma(e^{nt}-1)]\lambda(t)} \tag{3.8.1}$$

$$\dot{\lambda}(t) = \left(1 - \frac{\alpha}{s}\right)\left(a + \mu + \frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}\right)\lambda(t) \tag{3.8.2}$$

$$\Rightarrow \qquad \dot{\hat{k}}(t) = \left(\frac{\alpha}{s} - 1\right)\left(a + \mu + \frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}\right)\hat{k}(t) + \psi(t) - \hat{c}(t). \tag{3.8.3}$$

for $$\hat{k}(0) = \hat{k}_0 > 0, \quad \hat{k}(T_f) = \hat{k}_{T_f} \geq 0 \tag{3.8.4}$$

and $$\hat{z}(t) \approx \frac{\alpha}{s}\left(a + \mu + \frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}\right)\hat{k}(t) = \varphi(t)\hat{k}(t). \tag{3.8.5}$$

3.9 Solution of the Hamilton-Pontryagin equations

From Equation (3.8.2), it follows immediately that

$$\lambda(t) = \frac{\lambda_0}{[1+\sigma(e^{nt}-1)]^{\left(1-\frac{\alpha}{s}\right)}} e^{\left(1-\frac{\alpha}{s}\right)(a+n+\mu)t} \tag{3.9.1}$$

$$\Rightarrow \qquad \hat{c}(t) = \frac{1}{\lambda_0[1+\sigma(e^{nt}-1)]^{\frac{\alpha}{s}}} e^{\left[\left(\frac{\alpha}{s}-1\right)(a+\mu)+\frac{\alpha}{s}n-\gamma\right]t} \tag{3.9.2}$$

and $$\dot{\hat{k}}(t) = \left(\frac{\alpha}{s} - 1\right)\left(a + \mu + \frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}\right)\hat{k}(t) + \psi(t) - \frac{e^{\left[\left(\frac{\alpha}{s}-1\right)(a+\mu)+\frac{\alpha}{s}n-\gamma\right]t}}{\lambda_0[1+\sigma(e^{nt}-1)]^{\frac{\alpha}{s}}} \tag{3.9.3}$$

$$\Rightarrow \qquad \hat{k}(t) = [1 + \sigma(e^{nt} - 1)]^{\left(1-\frac{\alpha}{s}\right)} e^{\left(\frac{\alpha}{s}-1\right)(a+\mu+n)t}\left[\hat{k}_0 + \zeta(t) - \frac{1}{\lambda_0}h(t)\right] \tag{3.9.4}$$

where $h(t) = \int_0^t \frac{e^{-(\gamma-n)\tau}d\tau}{1+\sigma(e^{n\tau}-1)}$ and $\zeta(t) = \int_0^t \psi(\tau)[1 + \sigma(e^{n\tau} - 1)]^{\left(\frac{\alpha}{s}-1\right)} e^{\left(1-\frac{\alpha}{s}\right)(a+\mu+n)\tau} d\tau$.

Putting in the terminal condition gives

$$\lambda(t) = \frac{h(T_f)}{[\hat{k}_0+\zeta(T_f)]e^{\left(\frac{\alpha}{s}-1\right)(a+n+\mu)(T_f)} - \hat{k}_{T_f}\left[1+\sigma\left(e^{nT_f}-1\right)\right]^{\left(\frac{\alpha}{s}-1\right)}} \cdot \frac{e^{\left(1-\frac{\alpha}{s}\right)(a+n+\mu)(t-T_f)}}{[1+\sigma(e^{nt}-1)]^{\left(1-\frac{\alpha}{s}\right)}}. \tag{3.9.5}$$

$$\hat{c}(t) = \frac{[\hat{k}_0+\zeta(T_f)]e^{\left(\frac{\alpha}{s}-1\right)(a+n+\mu)T_f} - \hat{k}_{T_f}\left[1+\sigma\left(e^{nT_f}-1\right)\right]^{\left(\frac{\alpha}{s}-1\right)}}{h(T_f)e^{(\gamma-n)t}} \cdot \frac{e^{\left(\frac{\alpha}{s}-1\right)(a+\mu+n)(t-T_f)}}{[1+\sigma(e^{nt}-1)]^{\frac{\alpha}{s}}} \tag{3.9.6}$$

$$\hat{k}(t) = [1 + \sigma(e^{nt} - 1)]^{\left(1-\frac{\alpha}{s}\right)} e^{\left(\frac{\alpha}{s}-1\right)(a+\mu+n)t} q(t) \tag{3.9.7}$$

and $$\hat{y}(t) = \psi(t) + \varphi(t)[1 + \sigma(e^{nt} - 1)]^{\left(1-\frac{\alpha}{s}\right)} e^{\left(\frac{\alpha}{s}-1\right)(a+\mu+n)t} q(t) \tag{3.9.8}$$

where $q(t) = \frac{\hat{k}_0\left(h(T_f)-h(t)\right)+h(T_f)\zeta(t)-\zeta(T_f)h(t)+\hat{k}_{T_f}\left[1+\sigma\left(e^{nT_f}-1\right)\right]^{\left(\frac{\alpha}{s}-1\right)} e^{\left(1-\frac{\alpha}{s}\right)(a+n+\mu)T_f} h(t)}{h(T_f)}$.

Hence, if A_0 is the initial value of $A(t)$, then from our substitutions, we recover

$$c(t) = \frac{A_0\left[[\hat{k}_0+\zeta(T_f)]e^{\left(\frac{\alpha}{s}-1\right)(a+n+\mu)T_f}-\hat{k}_{T_f}\left[1+\sigma(e^{nT_f}-1)\right]^{\left(\frac{\alpha}{s}-1\right)}\right]}{h(T_f)e^{(\gamma-n-a)t}} \cdot \frac{e^{\left(\frac{\alpha}{s}-1\right)(a+\mu+n)(t-T_f)}}{[1+\sigma(e^{nt}-1)]^{\frac{\alpha}{s}}} \tag{3.9.9}$$

$$k(t) = A_0[1+\sigma(e^{nt}-1)]^{\left(1-\frac{\alpha}{s}\right)}e^{\left[\left(\frac{\alpha}{s}-1\right)(\mu+n)+\frac{\alpha}{s}a\right]t}q(t) \tag{3.9.10}$$

and $$y(t) = A_0\left[\psi(t)+\varphi(t)[1+\sigma(e^{nt}-1)]^{\left(1-\frac{\alpha}{s}\right)}e^{\left(\frac{\alpha}{s}-1\right)(a+\mu+n)t}q(t)\right]e^{at}. \tag{3.9.11}$$

However, whenever equilibrium is reached, the trajectory of y assumes the form defined in Equation (3.6.5) thereafter. The trajectories of c and k follow similar traits.

Using technological processes defined by $A_1(t)$ or $A_2(t)$ or $A_3(t)$, we obtain similar sets of trajectories for $c(t)$, $k(t)$, and $y(t)$, similar in structure to those in Equations (3.9.9) to (3.9.11), except for some few details. Let $y_i(t)$ be the counterpart trajectory for $y(t)$ in respect of $A_i(t)$, for $i = 1, 2, 3$, then

$$y_1(t) = \left[\psi_1(t)+\varphi_1(t)e^{\left(\frac{\alpha}{s}-1\right)(a+\mu+n)t}q_1(t)\right]A_1(t) \tag{3.9.12}$$

$$y_2(t) = \left[\psi_2(t)+\varphi_2(t)[1+\sigma(e^{nt}-1)]^{\left(1-\frac{\alpha}{s}\right)\left(\frac{1+\phi-\theta}{1-\theta}\right)}e^{\left(\frac{\alpha}{s}-1\right)\left(\mu+n\frac{1+\phi-\theta}{1-\theta}\right)t}q_2(t)\right]A_2(t) \tag{3.9.13}$$

$$y_3(t) = \left[\psi_3(t)+\varphi_3(t)[\{1+\xi(e^{at}-1)\}\{1+\sigma(e^{nt}-1)\}]^{\left(1-\frac{\alpha}{s}\right)}e^{\left(\frac{\alpha}{s}-1\right)(a+\mu+n)t}q_3(t)\right]A_3(t) \tag{3.9.14}$$

$$\psi_1(t) = \rho(1-\alpha)\left(\frac{\rho s}{\mu+\delta}\right)^{\frac{\alpha}{1-\alpha}}, \psi_2(t) = \rho(1-\alpha)\left(\frac{\rho s}{\mu+\frac{1+\phi-\theta}{1-\theta}\cdot\frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}}\right)^{\frac{\alpha}{1-\alpha}},$$

$$\psi_3(t) = \rho(1-\alpha)\left(\frac{\rho s}{\mu+\frac{a(1-\xi)}{1+\xi(e^{at}-1)}+\frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}}\right)^{\frac{\alpha}{1-\alpha}},$$

$$\varphi_1(t) = \frac{\alpha}{s}(\mu+\delta), \varphi_2(t) = \frac{\alpha}{s}\left(\mu+\frac{1+\phi-\theta}{1-\theta}\cdot\frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}\right), \varphi_3(t) = \frac{\alpha}{s}\left(\mu+\frac{a(1-\xi)}{1+\xi(e^{at}-1)}+\frac{n(1-\sigma)}{1+\sigma(e^{nt}-1)}\right)$$

$$q_1(t) = \frac{\hat{k}_0\big(h(T_f)-h(t)\big)+h(T_f)\zeta_1(t)-\zeta_1(T_f)h(t)+\hat{k}_{T_f}e^{\left(1-\frac{\alpha}{s}\right)(\mu+\delta)T_f}h(t)}{h(T_f)}, \zeta_1(t) = \frac{1-\alpha}{s-\alpha}\left(\frac{\rho s}{\mu+\delta}\right)^{\frac{1}{1-\alpha}}\left(e^{\left(1-\frac{\alpha}{s}\right)(\mu+\delta)t}-1\right)$$

$$q_2(t) = \frac{\hat{k}_0\left(h(T_f)-h(t)\right)+h(T_f)\zeta_2(t)-\zeta_2(T_f)h(t)+\hat{k}_{T_f}[1+\sigma(e^{nT_f}-1)]^{\left(\frac{\alpha}{s}-1\right)\left(\frac{1+\phi-\theta}{1-\theta}\right)}e^{\left(1-\frac{\alpha}{s}\right)\left(\mu+n\frac{1+\phi-\theta}{1-\theta}\right)T_f}h(t)}{h(T_f)}$$

$$q_3(t) = \frac{\hat{k}_0\left(h(T_f)-h(t)\right)+h(T_f)\zeta_3(t)-\zeta_3(T_f)h(t)+\hat{k}_{T_f}[\{1+\xi(e^{aT_f}-1)\}\{1+\sigma(e^{nT_f}-1)\}]^{\left(\frac{\alpha}{s}-1\right)}e^{\left(1-\frac{\alpha}{s}\right)(a+\mu+n)T_f}h(t)}{h(T_f)}$$

$$\zeta_2(t) = \int_0^{T_f}\psi_2(\tau)[1+\sigma(e^{n\tau}-1)]^{\left(\frac{\alpha}{s}-1\right)\left(\frac{1+\phi-\theta}{1-\theta}\right)\tau}e^{\left(1-\frac{\alpha}{s}\right)\left(\mu+n\frac{1+\phi-\theta}{1-\theta}\right)\tau}d\tau$$

$$\zeta_3(t) = \int_0^{T_f}\psi_3(\tau)[\{1+\xi(e^{a\tau}-1)\}\{1+\sigma(e^{n\tau}-1)\}]^{\left(\frac{\alpha}{s}-1\right)}e^{\left(1-\frac{\alpha}{s}\right)(a+\mu+n)\tau}d\tau.$$

In each case, the equilibrium trajectory of $y_i(t)$, for $i = 1, 2, 3$, is given by

$$y_i^*(t) = \frac{1}{1-\alpha}\psi_i(t)A_i(t). \tag{3.9.15}$$

4 Discussion

4.1 Models with constant technological growth

From the above, it is obvious that the equilibrium value of real per capita GDP, $y(t)$, is higher in a system with logistic labour growth ($\sigma > 0$) than one in which labour growth is exponential ($\sigma = 0$). Equally, as per the sensitivity and numerical simulations, it is evident that the former economy grows faster, and given that it generates higher equilibrium, establishes higher time-values of $y(t)$ than the latter, for any set of parameter values. That is, for all $t \geq 0, \frac{\partial y}{\partial \sigma} > 0$. All the simulation plots, especially Fig. 4.1 and Fig. 4.2, confirm this. The population dynamics is indicative of these conclusions. (Data used has been sourced from [38], but the analysis stops short of linking up to the actual economies, especially as simulations are performed on these parameter values to check for qualitative changes.[5])

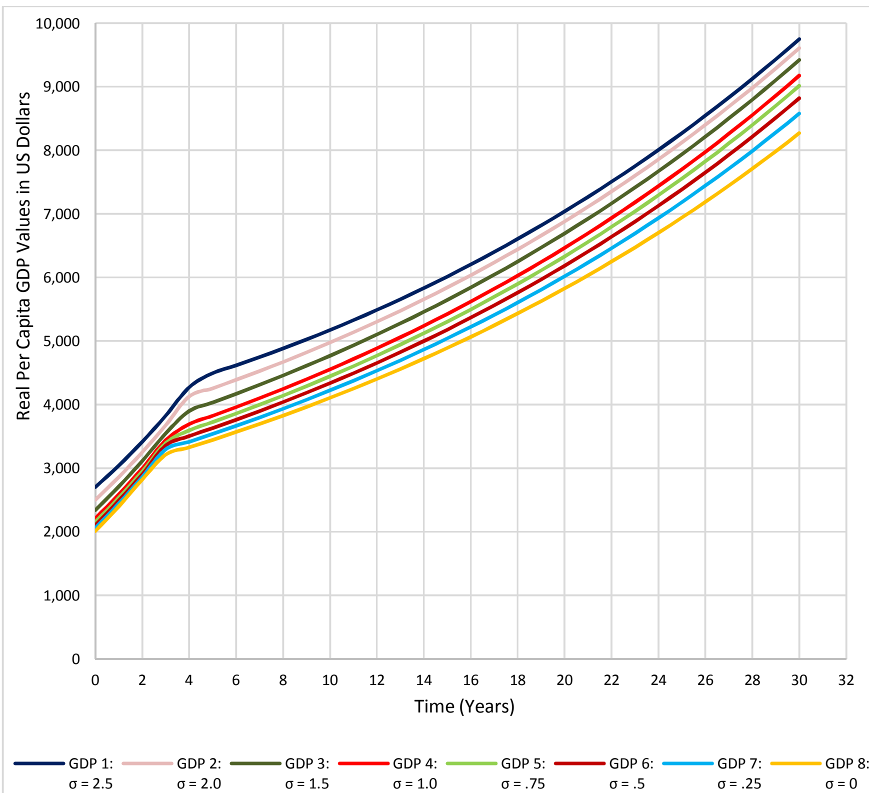

Fig. 4.1. Real per capita GDP trajectories for varying values of σ (With Tech)

[5] *For any clearly defined bundle of trajectories, the lower trajectory depicts the case where $\sigma = 0$ (population growing exponentially), with increasingly higher trajectories as σ steadily becomes higher (population progressively becomes logistic). From one bundle of trajectories to another which is higher shows the application of increasing value of a parameter (besides σ) on which the simulation is being carried out. This does not happen when R & D technology is at play.*

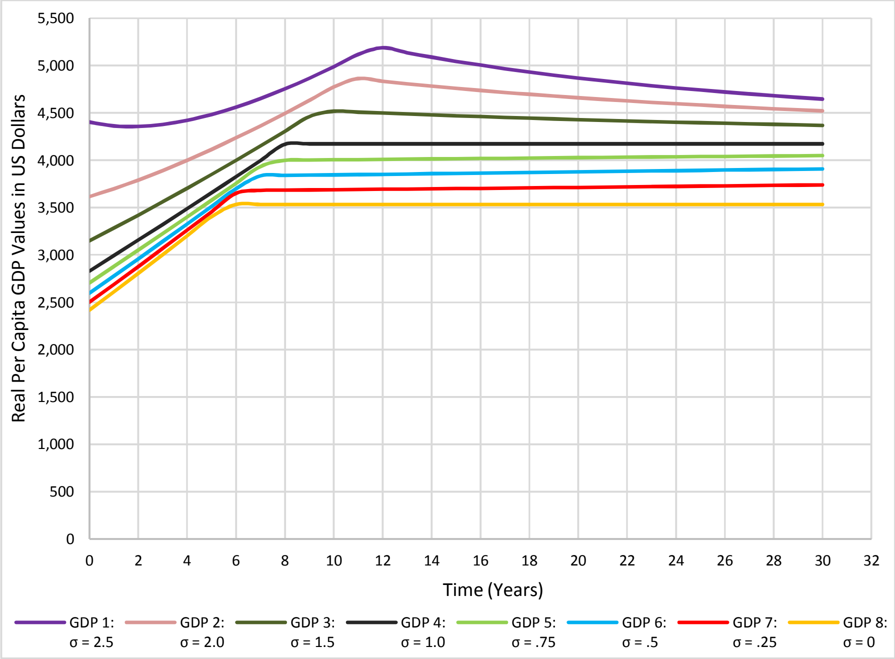

Fig. 4.2. Real per capita GDP trajectories for varying values of σ (No Tech)

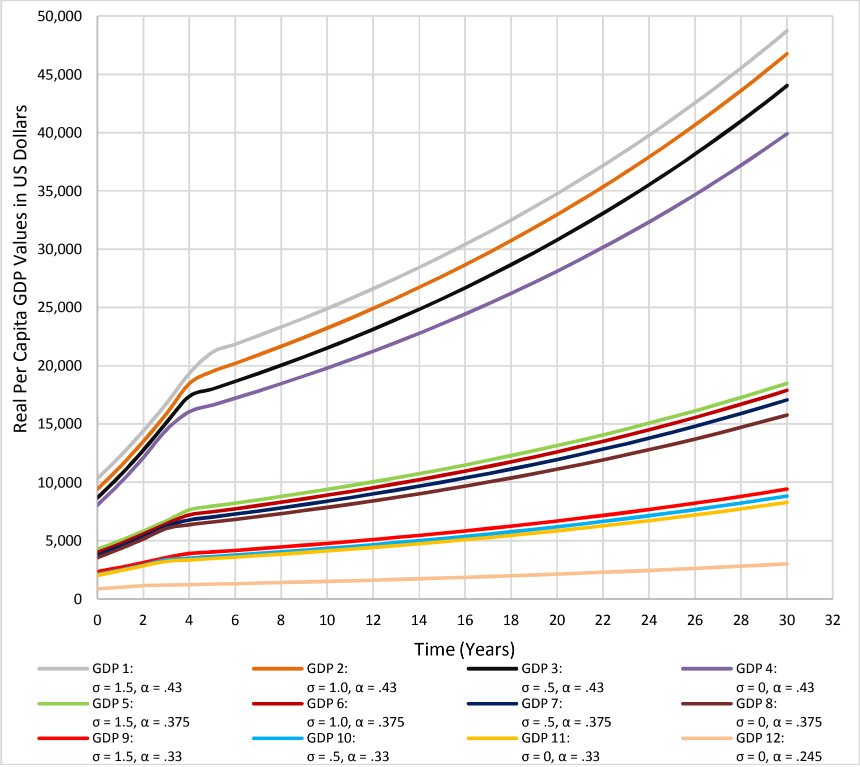

Fig. 4.3. Real per capita GDP trajectories for varying values of σ and α (With Tech)

Like noted in [8], the results suggest $\frac{\partial y}{\partial a} > 0$, and that higher values of a excite higher time-performance in $y(t)$, and vice versa. All the simulation plots, especially Fig. 4.9, re-enforce this finding. The trajectory of $y(t)$ takes a nose-dive when $a < 0$.[6] The real per capita income trajectory, $y(t)$, rises gently over time, almost flat, whenever $a = 0$, and $0 < \sigma < 1$, but declines asymptotically when $\sigma > 1$. But $y(t)$ is either completely flat or it may initially rise and thereafter flatten up, when $a = 0$, and $\sigma = 0$.

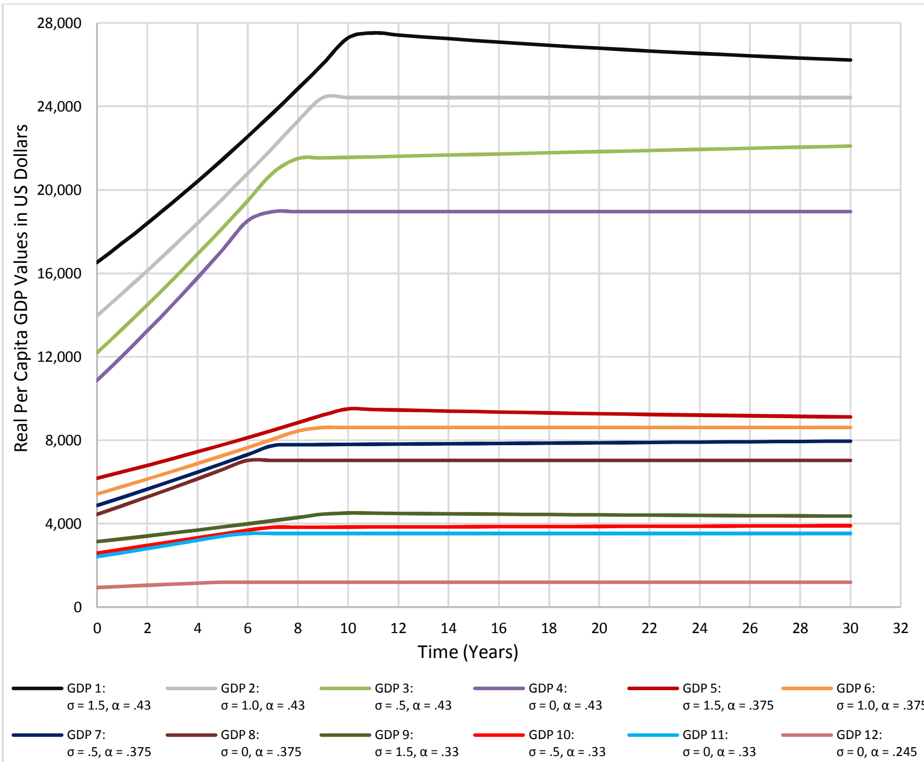

Fig. 4.4. Real per capita GDP trajectories for varying values of σ and α (No Tech)

Similarly, it can be inferred that $\frac{\partial y}{\partial \alpha} > 0$, whenever $0 < \sigma < 1$, but $\frac{\partial y}{\partial \alpha} < 0$ when $\sigma > 1$, and $\frac{\partial y}{\partial \alpha} = 0$ when $\sigma = 0, 1$. Fig. 4.3 and Fig. 4.4 above, as well as Fig. 4.6 and Fig. 4.7 below, lend a lot of credence to this. It can also be confirm that $\frac{\partial y}{\partial s} > 0$. Fig. 4.5 and Fig. 4.6 beneath attest to this. (This is also true in respect of ρ. Also, $\frac{\partial y}{\partial \hat{k}_0} = 0$, confirming the fact that $\hat{k}$ tends to a finite limit, $\hat{k}_{e_2}$, for any $\hat{k}_0 > 0$.) Nonetheless, the growth potential in $y(t)$ arising out of α naturally surpasses that of s.

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[6] *In the referenced plots above, unless otherwise stated, we have largely used* $\rho = 150$, $\alpha = 0.33$, $s = 0.285$, $a = 0.035$, $n = 0.02$, $\hat{k}_0 = 600$,
$\gamma = 0.045$, $\mu = 0.05$, $A_0 = 1$, $\phi = 0.8$, $\theta = 0.6$, $\delta = 0.042$ *and* $\xi = 0.05$.

$\sigma = 0$	*the population (labour) grows exponentially at its natural growth rate n over time.*
$\sigma = 1$	*the population (labour) will be static and not growing at all or growing at a 0.00% over time.*
$\sigma > 1$	*the population (labour) will be declining or growing at a negative rate over time.*
$0 < \sigma < 1$	*the population (labour) will be growing but at a reducing rate over time.*

For $\sigma > 0$, *the population growth dynamics is logistic. We assume that the natural growth rate of population,* n, *is* $0 < n < 1$.

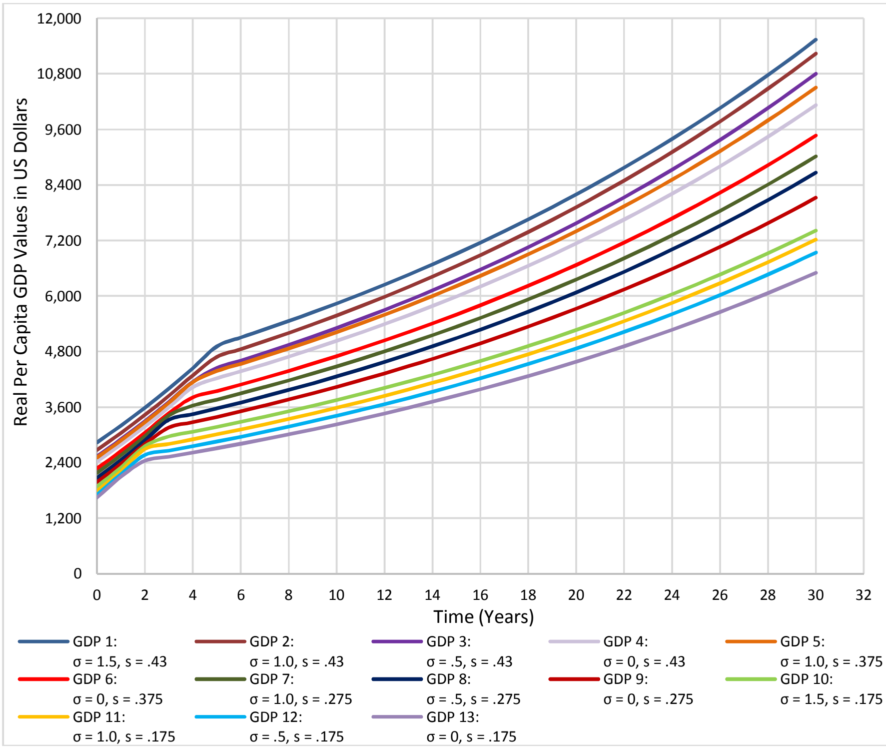

Fig. 4.5. Real per capita GDP trajectories for varying values of σ and s (With Tech)

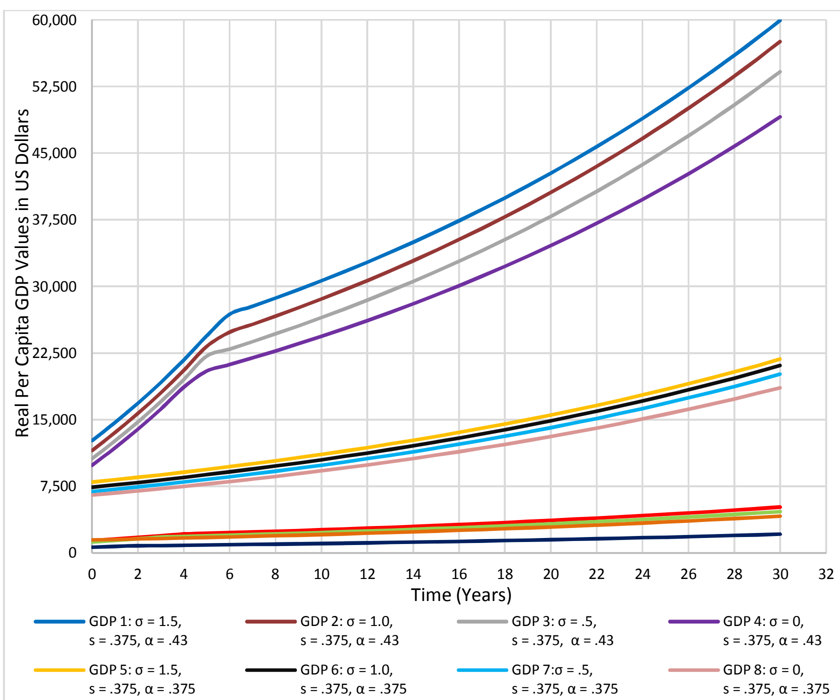

Fig. 4.6. Real per capita GDP trajectories for varying values of σ, α and s (With Tech)

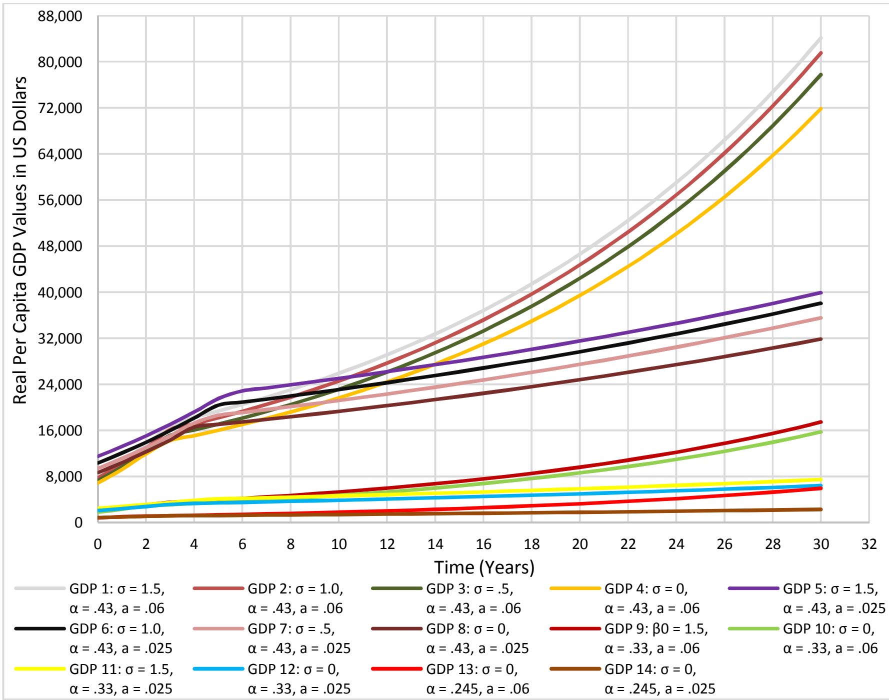

Fig. 4.7. Real per capita GDP trajectories for varying values of σ, α and a

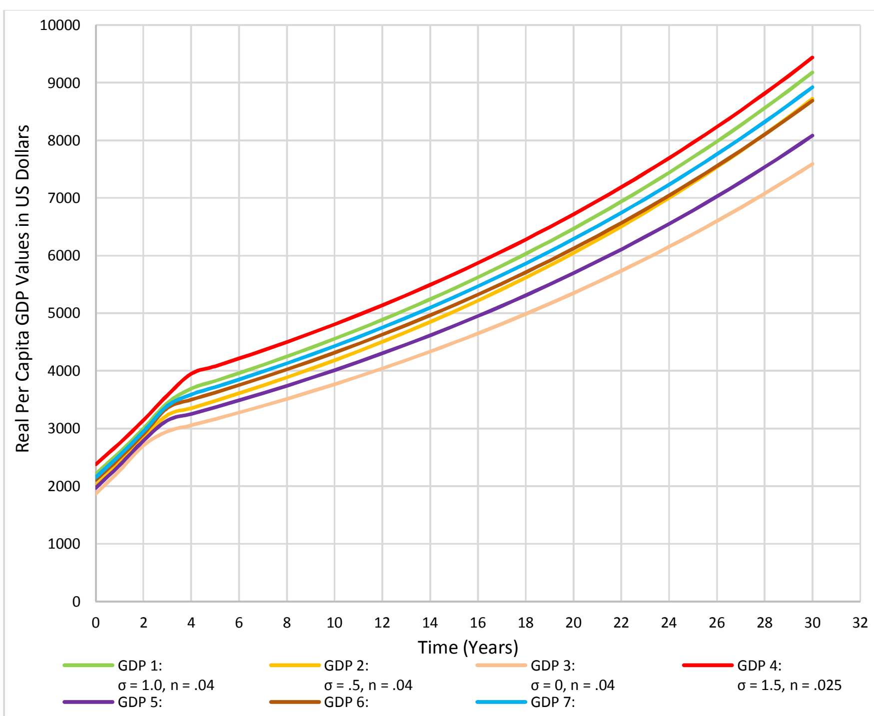

Fig. 4.8. Real per capita GDP trajectories for varying values of σ and n (With Tech)

It can be confirmed that $\frac{\partial y}{\partial n} < 0$, for all $t \geq 0$, when $\sigma = 0$ or $\sigma > 1$. Similarly, $\frac{\partial y}{\partial n} > 0$ when $0 < \sigma < 1$, whereas $\frac{\partial y}{\partial n} = 0$ whenever $\sigma = 1$. Fig. 4.8 attest to these. (From the simulations and Equations (3.6.10), (3.6.6) and confirms that $\frac{\partial y}{\partial \mu} < 0$, for all $t \geq 0$. Unlike in [8], however, $\frac{\partial y}{\partial \gamma} = 0$.)

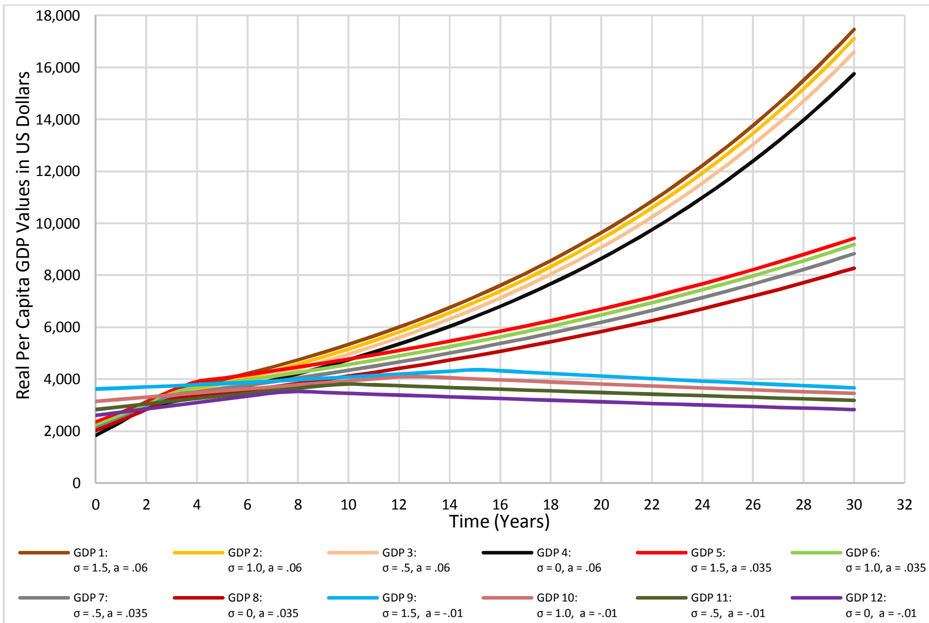

Fig. 4.9. Real per capita GDP trajectories for varying values of σ and a

4.2 Systems with non-constant technological growth dynamics

The contrast between these technological processes and the other with constant growth is that either the associated population growth here is non-constant or the technological growth dynamics itself is logistic. Here, the effects of s and α (as well as those of ρ, $\hat{k}_0$, γ and μ) are largely the same as discussed earlier. As suspected earlier in Section 3.2, the lower the value of n, the higher the growth prospects, and hence, the time-performance of $y(t)$ in the system with residual technological progress, $A_1(t)$, and vice versa. This is similarly true in respect of systems underpinned by logistic technological process, $A_3(t)$,[7] except that the negative effect here is far less than in the former, all things being equal. The reverse is the truth when the technological process is premised on the R & D phenomenon, $A_2(t)$.

Unlike in the simplistic form in which technological growth is constant, in systems underlain by R & D, the performance of $y(t)$ is highest when $\sigma = 0$, and drastically slows down when $\sigma > 0$. The growth prospects in $y(t)$ here is much enhanced for higher values of ϕ and θ, especially the latter. However, the effect of σ on systems with technological processes $A_1(t)$ and $A_3(t)$ is similar in nature to its effect on the systems with constant technological growth. But the system with the technological process $A_3(t)$ generates the highest time-trajectory of $y(t)$ when $\xi = 0$, worsening with increasing values of ξ. This highest growth rate generated by $A_3(t)$ is at a constant level a, same as used in the original models. Fig. 4.10 and Fig. 4.11 below shows the effects of $A_i(t)$ on the performance of $y(t)$.

In systems with technological process $A_1(t)$, $y(t)$ grows much faster at a constant rate δ, when $n = 0$ or $\sigma = 1$, and much more (no longer constant) when $\sigma > 0$ and $0 < n < 1$, except that $\sigma > 1$ is not desirable for population sustenance. Anytime $\delta - n > 0$, the time-performance of $y(t)$ is good.

[7] *The logistic technological process is indicated in the plot as L'Tech.*

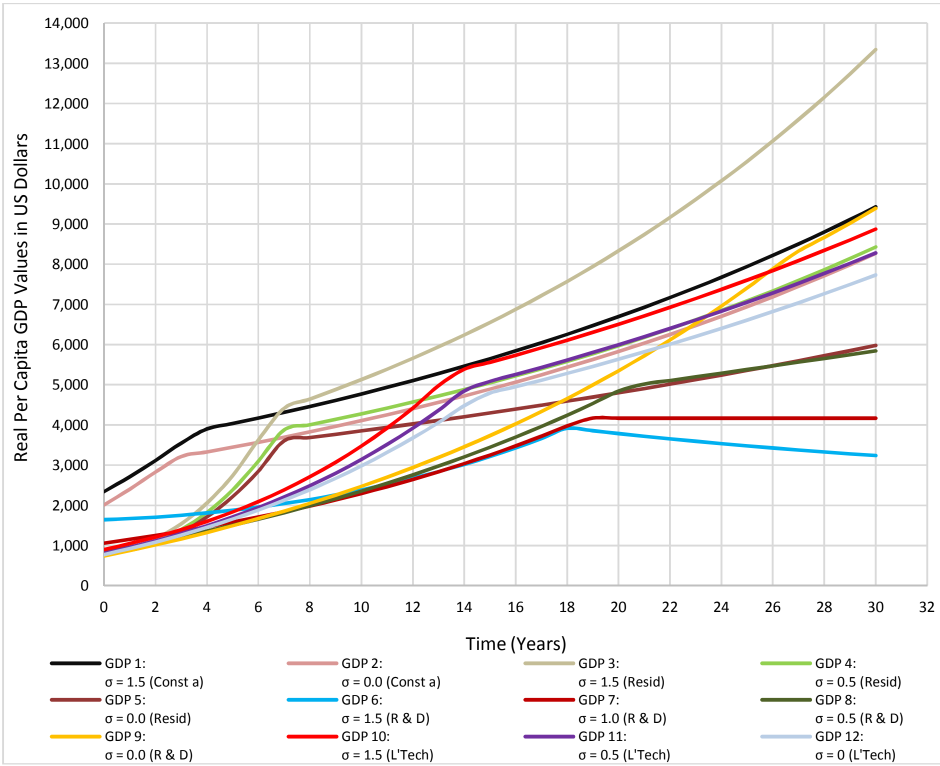

Fig. 4.10. Real per capita GDP trajectories for various Tech processes and σ values

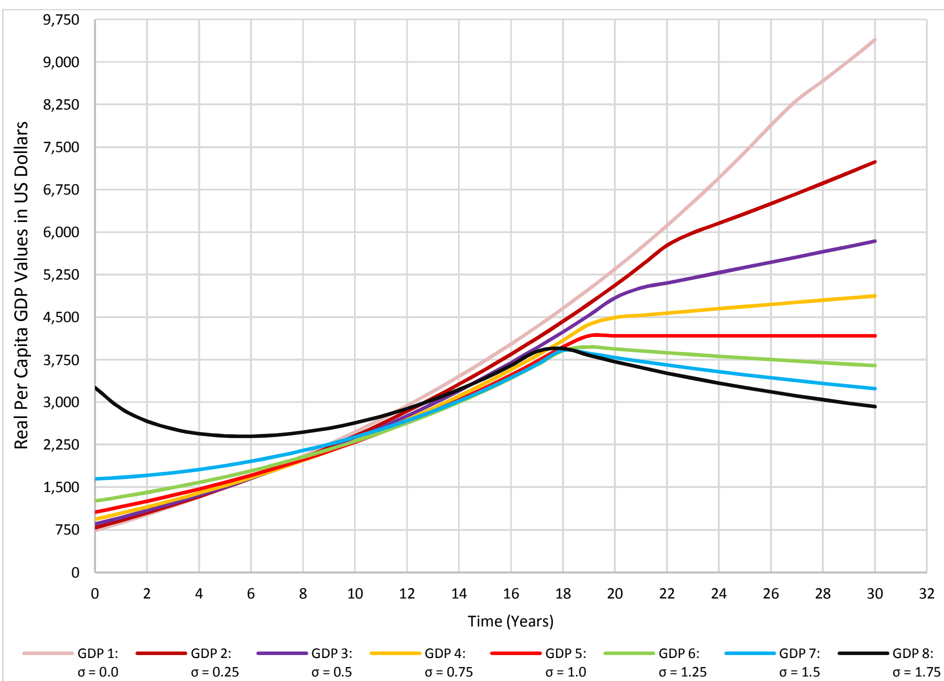

Fig. 4.11. Real per capita GDP trajectories for various σ values Under R & D Tech process

5 Conclusion

The models constructed in the foregoing are generally stable, each in the neighbourhood of its non-trivial equilibrium value, whenever $0 \leq \sigma \leq 1$. They are each locally controllable and observable, and thus, stabilizable and detectable. Consequently, the models' solutions are feasible and reachable, and most importantly, bounded inputs always generate bounded outputs. Thus as expected, their predictions are plausible and realistic.

Furthermore, it is also found from the comparative analyses of the results that:

1) Under the framework of R & D technology, economies with exponential population growth consistently perform better than those with logistic population growth.[8]
2) In contrast to the above, it is also clear that, under other sets of conditions, excluding the R & D technological process, economies with exponential population growth consistently perform worse than those underpinned by logistic population growth.
3) Higher technological growth is also found to be an excellent tool for rapid economic growth, and with this, it is clear from the simulation graphs that a lower income economy, over a time, can surpass that of a higher income economy whose technological growth is much less.
4) The population dynamics parameter, σ, largely dictates, how most of the other parameters, and the technological processes, impact on the time-performance of real per capita income.
5) Given the caveat on σ, its tolerable domain is $0 \leq \sigma \leq 1$. The border line value $\sigma = 1$ may not be advisable given the caution on σ.
6) But $\sigma = 0$ is appropriate for high time-performance of real per capita income if and only if the R & D technology is in place.
7) However, $\sigma = 1$ is ideal when the modified residual technological process is rather in place.
8) Generally, $0 < \sigma \leq 1$ is preferable for high time-performance of real per capita income, except with the residual technology where $\frac{\partial y}{\partial n} < 0$, whenever $0 \leq \sigma < 1$.
9) Additionally, programmes that enhance technology, savings and marginal products of the various factors of production, per (effective) labour, whilst ensuring lower factor depreciation rates, and minimal shares of natural resource-based factors in the aggregate production function, help exit the Malthusian trap, if it exists, or postpone indefinitely.

Competing Interests

Authors have declared that no competing interests exist.

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[8] The Asian tigers (e.g., South Korea, Hong Kong, Singapore), as well as China and Malaysia may exemplify those economies that may have turned the high population growth rate or densities, prior to the period of sustained growth, into positive catalytic benefits through education and skill training. The resulting technologies developed and or applied quickened the transformation of these classes of economies into high income and higher middle income economies respectively in no time, and most of these economies are still growing that fast. In most developed economies (before the advent the above mentioned ones) like Belgium, Sweden, Britain, Norway, whereas population growth is logistic, economic growth is not as rapid as the former set of economies mentioned.

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