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# An Algorithm for Solutions of Hammerstein Integral Equations with Maximal Monotone Operators in Classical Banach Spaces

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***Authors' contributions***

*This work was carried out in collaboration between both authors. Both authors made significant contribution, author MOU wrote the first draft of the manuscript. All authors read and approved the final manuscript.*

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## Abstract

Let $X = l_p$ , $2 \leq p < \infty$. Let $F : X \to X^*$ and $K : X^* \to X$ be bounded maximal monotone mappings such that the Hammerstein equation $u + KFu = 0$ has a solution. An explicit iteration sequence is constructed and proved to converge strongly to a solution of this equation. Our method of proof is also of independent interest.

*Keywords: Bounded maximal monotone mappings; hammerstein equations; strong convergence.*

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# 1 Introduction

Let $H$ be a real Hilbert space. A map $A : H \to 2^H$ is called *monotone* if for each $x, y \in H$, the following inequality holds:

$$\langle \xi - \tau, x - y \rangle \geq 0 \ \ \forall \ \xi \in Ax, \tau \in Ay. \tag{1.1}$$

The monotonicity condition in Hilbert space has been extended to arbitrary normed linear spaces. To introduce one of two known and studied extensions, we need the following definition.
Let $E$ be a real normed space with dual space $E^*$. A mapping $J : E \to 2^{E^*}$ is called the *normalised duality map* if for each $x \in E$,

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = ||x|| ||x^*||, ||x|| = ||x^*||\}.$$

A map $A : E \to 2^E$ is called *accretive* if for each $x, y \in E$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle \xi - \tau, j(x-y) \rangle \geq 0, \ \ \forall \ \xi \in Ax, \tau \in Ay. \tag{1.2}$$

It is well known that if $E$ is a real Hilbert space, then $J = I$, the identity map on $E$. In this case, the inequality (1.2) reduces to inequality (1.1). Hence, accretivity in normed spaces is one extension of Hilbert space monotonicity condition to arbitrary real normed spaces.

Also, a mapping $A : E \to 2^{E^*}$ is called *monotone* if for all $x, y \in D(A)$

$$\langle \xi - \zeta, x - y \rangle \geq 0 \ \ \forall \xi \in Ax, \ \forall \zeta \in Ay. \tag{1.3}$$

It is clear that if $E = H$ a real Hilbert space, then $E = E^* = H$ and inequality (1.3) coincides with the monotonicity definition in Hilbert spaces. So, this is another extension of Hilbert space monotonicity.

A mapping $A : X \to 2^{X^*}$ is said to be maximal monotone if it is monotone and for $(x, u) \in X \times X^*$ the inequalities $\langle u - v, x - y \rangle \geq 0$, for all $(y, v) \in G(A)$, imply $(x, u) \in G(A)$ where $G(A)$ is the graph of $A$.

Let $\Omega \subset \mathbb{R}^n$ be bounded. Let $k : \Omega \times \Omega \to \mathbb{R}$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be measurable real-valued functions. An integral equation (generally nonlinear) of *Hammerstein-type* has the form

$$u(x) + \int_\Omega k(x, y) f(y, u(y)) dy = w(x), \tag{1.4}$$

where the unknown function $u$ and inhomogeneous function $w$ lie in a Banach space $E$ of measurable real-valued functions. If we define $F : \mathcal{F}(\Omega, \mathbb{R}) \to \mathcal{F}(\Omega, \mathbb{R})$ and $K : \mathcal{F}(\Omega, \mathbb{R}) \to \mathcal{F}(\Omega, \mathbb{R})$ by

$$Fu(y) = f(y, u(y)), \ x \in \Omega,$$

and

$$Kv(x) = \int_\Omega k(x, y) v(y) dy, \ x \in \Omega,$$

respectively, where $\mathcal{F}(\Omega, \mathbb{R})$ is a space of measurable real-valued functions defined from $\Omega$ to $\mathbb{R}$, then equation (1.4) can be put in the abstract form

$$u + KFu = 0. \tag{1.5}$$

where, without loss of generality, we have assumed that $w \equiv 0$.

Interest in (1.4) stems mainly from the fact that several problems that arise in differential equations, for instance, elliptic boundary value problems whose linear parts possess Green's function can, as a rule, be transformed into the form (1.4) (see e.g., Pascali and Sburian [1], chapter IV, p. 164).

Equations of Hammerstein-type also play a crucial role in the theory of optimal control systems and in automation and network theory (see e.g., Dolezale [2]).

Several existence results have been proved for equations of Hammerstein-type (see e.g., Brézis and Browder [3, 4, 5], Browder [6], De Figueiredo and Gupta [7]).

In general, equations of Hammerstein-type are nonlinear and there is no known method to find close form solutions for them. Consequently, methods of approximating solutions of such equations, where solutions are known to exist, are of interest. Attempts had been made to approximate solutions of equations of Hammerstein-type using *Mann-type* (see e.g., Mann [8]) iteration scheme. However, the results obtained were not satisfactory (see [9]). The recurrence formulas used in these attempts, even in real Hilbert spaces, involved $K^{-1}$ which is required to be strongly monotone when $K$ is, and this, apart from limiting the class of mappings to which such iterative schemes are applicable, is also not convenient in any possible applications.

Part of the difficulty in establishing *iterative algorithms* for approximating solutions of Hammerstein equations seems to be that the composition of two monotone maps need not be monotone.

The first satisfactory results on *iterative methods* for approximating solutions of Hammerstein equations *involving accretive-type mappings*, as far as we know, were obtained by Chidume and Zegeye [10, 11].

Recently, the following important result was proved in a Hilbert space by Chidume and Djitte.

**Theorem 1.1.** *[Chidume and Djitte, [12]]Let $H$ be a real Hilbert space and $F, K : H \to H$ be bounded and maximal monotone operators. Let $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ be sequences in $H$ defined iteratively from arbitrary points $u_1,\ v_1 \in H$ as follows:*

$$u_{n+1} = u_n - \lambda_n(Fu_n - v_n) - \lambda_n\theta_n(u_n - u_1),\ n \geq 1, \tag{1.6}$$

$$v_{n+1} = v_n - \lambda_n(Kv_n + u_n) - \lambda_n\theta_n(v_n - v_1),\ n \geq 1, \tag{1.7}$$

*where $\{\lambda_n\}_{n=1}^{\infty}$ and $\{\theta_n\}_{n=1}^{\infty}$ are sequences in $(0, 1)$ satisfying the folliowing conditions:*
*(i)* $\lim_{n\to\infty} \theta_n = 0$,
*(ii)* $\sum_{n=1}^{\infty} \lambda_n\theta_n = \infty, \quad \lambda_n = o(\theta_n)$,
*(iii)* $\lim_{n\to\infty} \frac{\left(\frac{\theta_{n-1}}{\theta_n} - 1\right)}{\lambda_n\theta_n} = 0.$

*Suppose that $u + KFu = 0$ has a solution in $H$. Then, there exists a constant $d_0 > 0$ such that if $\lambda_n \leq d_0\theta_n$ for all $n \geq n_0$ for some $n_0 \geq 1$, then the sequence $\{u_n\}_{n=1}^{\infty}$ converges to $u^*$, a solution of $u + KFu = 0$.*

*Remark* 1.1. As is well known, among all infinite dimensional Banach spaces, Hilbert spaces have the *nicest* geometric properties (Chidume [13]). However, even with these nice properties of Hilbert spaces, it is known that many and probably most, mathematical objects and models do not naturally live in Hilbert spaces. We quickly remark that once one moves out of Hilbert spaces, one loses these nice properties.

It is our purpose in this paper to prove a strong convergence theorem in $l_p$ spaces $(2 \leq p < \infty)$ which extends Theorem 1.1 to a more general space. Our method of proof is also of independent interest.

# 2 Preliminaries

Let $E$ be a normed space with $dimE \geq 2$. The *modulus of convexity* of $E$ is the function $\delta_E : (0,2] \to [0,1]$, defined by

$$\delta_E(\epsilon) := inf\Big\{1 - \Big\|\frac{x+y}{2}\Big\| : \|x\| = \|y\| = 1; \epsilon = \|x-y\|\Big\}.$$

It is known (see e.g., Alber [14]) that if

- $E = l_p$, $(2 \leq p < \infty)$ then
$$\delta_E(\epsilon) \geq p^{-1}\Big(\frac{\epsilon}{2}\Big)^p \tag{2.1}$$
- $E = l_p$, $(1 < p \leq 2)$ then
$$\delta_E(\epsilon) \geq \frac{(p-1)\epsilon^2}{16} \tag{2.2}$$
- $E = l_q$, $(1 < q \leq 2)$ then
$$\delta_E(\epsilon) \geq \frac{(q-1)\epsilon^2}{16} \tag{2.3}$$

The space $E$ is *uniformly convex* if and only if $\delta_E(\epsilon) > 0$ for every $\epsilon \in (0,2]$.

A Banach space $E$ is said to be *strictly convex* if

$$\|x\| = \|y\| = 1, \quad x \neq y \implies \Big\|\frac{x+y}{2}\Big\| < 1.$$

Let $E$ be a real normed linear space of dimension $\geq 2$. The *modulus of smoothness* of $E$ , $\rho_E : [0,\infty) \to [0,\infty)$, is defined by:

$$\rho_E(\tau) := \sup\left\{\frac{\|x+y\| + \|x-y\|}{2} - 1 : \|x\| = 1, \|y\| = \tau, \quad \tau > 0\right\}.$$

A normed linear space $E$ is called *uniformly smooth* if

$$\lim_{\tau \to 0} \frac{\rho_E(\tau)}{\tau} = 0.$$

The norm of $E$ is said to be *Fréchet differentiable* if for each $x \in S := \{u \in E : \|u\| = 1\}$,

$$\lim_{t \to 0} \frac{\|x+ty\| - \|x\|}{t},$$

exists and is attained uniformly for $y \in E$.

In the sequel, we shall need the following definitions and results. Let $E$ be a smooth real Banach space with dual $E^*$. The function $\phi : E \times E \to \mathbb{R}$, defined by,

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad \text{for } x, y \in E, \tag{2.4}$$

where $J$ is the normalized duality mapping from $E$ into $2^{E^*}$. It was introduced by Alber and has been studied by Alber [15], Alber and Guerre-Delabriere [16], Kamimura and Takahashi [17], Reich [18] and a host of other authors. If $E = H$, a real Hilbert space, then equation (2.4) reduces to $\phi(x,y) = \|x-y\|^2$ for $x, y \in H$. It is obvious from the definition of the function $\phi$ that

$$(\|x\| - \|y\|)^2 \leq \phi(x,y) \leq (\|x\| + \|y\|)^2 \quad \text{for } x, y \in E. \tag{2.5}$$

Define a map $V : X \times X^* \to \mathbb{R}$ by

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^*\rangle + \|x^*\|^2. \tag{2.6}$$

Then, it is easy to see that

$$V(x, x^*) = \phi(x, J^{-1}(x^*)) \ \forall x \in X, \ x^* \in X^*. \tag{2.7}$$

**Lemma 2.1.** *([Alber, [15]]) Let $X$ be a reflexive striclty convex and smooth Banach space with $X^*$ as its dual. Then,*

$$V(x, x^*) + 2\langle J^{-1}x^* - x, y^*\rangle \leq V(x, x^* + y^*) \tag{2.8}$$

*for all $x \in X$ and $x^*, y^* \in X^*$.*

**Lemma 2.2** (Kamimura and Takahashi, [17])**.** *Let $X$ be a real smooth and uniformly convex Banach space, and let $\{x_n\}$ and $\{y_n\}$ be two sequences of $X$. If either $\{x_n\}$ or $\{y_n\}$ is bounded and $\phi(x_n, y_n) \to 0$ as $n \to \infty$, then $\|x_n - y_n\| \to 0$ as $n \to \infty$.*

**Lemma 2.3.** *(Xu [19]) Let $\rho_n$ be a sequence of non-negative real numbers satisfying the relation:*

$$\rho_{n+1} \leq (1 - \beta_n)\rho_n + \beta_n\zeta_n + \gamma_n, \ n \geq 0, \tag{2.9}$$

*where,*

*(i) $\beta_n \subset [0, 1]$, $\sum \beta_n = \infty$; (ii) $\limsup \zeta_n \leq 0$; (iii) $\gamma_n \geq 0$; $(n \geq 0)$, $\sum \gamma_n < \infty$. Then, $\rho_n \to 0$ as $n \to \infty$.*

*Remark* 2.1. Let $E^*$ be a strictly convex dual Banach space with a Fréchet differentiable norm and $A : E \to 2^{E^*}$, be a maximal monotone map with no monotone extension. Let $z \in E^*$ be fixed. Then for every $\lambda > 0$, there exists a unique $x_\lambda \in E$ such that $z \in Jx_\lambda + \lambda Ax_\lambda$ (see Reich [20], p. 342). Setting $J_\lambda z = x_\lambda$, we have the resolvent $J_\lambda := (J + \lambda A)^{-1} : E^* \to E$ of $A$, for every $\lambda > 0$. A celebrated result of Reich follows.

**Lemma 2.4.** *(Reich, [20]). Let $E^*$ be a strictly convex dual Banach space with a Fréchet differentiable norm and let $A : E \to E^*$ be maximal monotone such that $A^{-1}0 \neq \emptyset$. Let $z \in E^*$ be an arbitrary but fixed vector. For each $\lambda > 0$, there exists a unique $x_\lambda \in E$ such that $z \in Jx_\lambda + \lambda Ax_\lambda$. Furthermore, $x_\lambda$ converges strongly to a unique $v \in A^{-1}0$.*

**Lemma 2.5** (Alber, [14], p45)**.** *Let $X$ be a uniformly convex Banach space. Then for any $R > 0$ and any $x, y \in X$ such that $\|x\| \leq R$, $\|y\| \leq R$ the following inequality holds:*

$$\langle Jx - Jy, x - y\rangle \geq (2L)^{-1}\delta_X(c_2^{-1}\|x - y\|), \tag{2.10}$$

*where $c_2 = 2\max\{1, R\}$, $1 < L < 1.7$.*

For $X = l_p$, $(2 \leq p < \infty)$ we have that

$$\langle Jx - Jy, x - y\rangle \geq \frac{\|x - y\|^p}{2^{p+1}pLc_2^p}. \tag{2.11}$$

Also for $X = l_q$, $(1 < q \leq 2)$ we have using 2.3 that

$$\langle Jx - Jy, x - y\rangle \geq \frac{(q - 1)\|x - y\|^2}{32Lc_2^2}. \tag{2.12}$$

**Lemma 2.6** (Alber, [14], p45). *Let $X$ be a uniformly smooth and strictly convex Banach space. Then for any $R > 0$ and any $x$, $y \in X$ such that $\|x\| \leq R$, $\|y\| \leq R$ the following inequality holds:*

$$\langle Jx - Jy, x - y\rangle \geq (2L)^{-1}\delta_{X^*}(c_2^{-1}\|Jx - Jy\|), \tag{2.13}$$

*where $c_2 = 2\max\{1, R\}$, $1 < L < 1.7$.*

For $X = l_p$, $(2 \leq p < \infty)$ we have using 2.3 that

$$\langle Jx - Jy, x - y\rangle \geq \frac{(q-1)\|Jx - Jy\|^2}{32Lc_2^2}. \tag{2.14}$$

Also, for $X = l_q$, $(1 < q \leq 2)$ we have using 2.3 that

$$\langle Jx - Jy, x - y\rangle \geq \frac{\|Jx - Jy\|^p}{2^{p+1}pLc_2^p}. \tag{2.15}$$

**Lemma 2.7** (Alber, [14], p.50). *Let $X$ be a reflexive strictly convex and smooth Banach space with dual $X^*$. Let $W : X \times X \to \mathbb{R}$ be defined by $W(x, y) = \frac{1}{2}\phi(y, x)$. Then,*

$$\phi(y, x) - \phi(y, z) \geq 2\langle Jx - Jz, z - y\rangle, \tag{2.16}$$

*and*

$$W(x, y) \leq \langle Jx - Jy, x - y\rangle, \tag{2.17}$$

*for all $x, y, z \in X$*

**Lemma 2.8.** *From Lemma 2.4, if we set $\lambda_n := \frac{1}{\theta_n}$ where $\theta_n \to 0$ as $n \to \infty$, $z = Jv$ for some $v \in E$, and $y_n := \left(J + \frac{1}{\theta_n}A\right)^{-1} z$, we obtain that:*

$$\begin{aligned} Ay_n &= \theta_n(Jv - Jy_n), \\ y_n &\to y^* \in A^{-1}0, \end{aligned} \tag{2.18}$$

*where $A : E \to E^*$ is maximal monotone. We observe that equation (2.18) yields*

$$Jy_{n-1} - Jy_n + \frac{1}{\theta_n}\Big(Ay_{n-1} - Ay_n\Big) = \frac{\theta_{n-1} - \theta_n}{\theta_n}\Big(Ju - Jy_{n-1}\Big).$$

*Taking the duality pairing of the LHS of this equation with $y_{n-1} - y_n$, and using the monotonicity of $A$ we obtain that,*

$$\langle Jy_{n-1} - Jy_n, y_{n-1} - y_n\rangle \quad \leq \quad \frac{\theta_{n-1} - \theta_n}{\theta_n}\Big\|Jv - Jy_{n-1}\Big\|\|y_{n-1} - y_n\|. \tag{2.19}$$

*It follows that for $E = l_p$ $(2 \leq p < \infty)$, using equations 2.11, 2.14 and 2.19 we obtain that,*

$$\|y_{n-1} - y_n\| \leq \left(\frac{\theta_{n-1} - \theta_n}{\theta_n}\right)^{\frac{1}{p}} C_1, \text{ and} \tag{2.20}$$

$$\|Jy_{n-1} - Jy_n\| \leq \left(\frac{\theta_{n-1} - \theta_n}{\theta_n}\right)^{\frac{1}{2}} C_2, \tag{2.21}$$

*where $C_1$ and $C_2$ are some positive constants.*
*Similarly, for $E^* = l_q$ $(1 < q \leq 2)$, using equations 2.12, 2.15 and 2.19 we obtain that,*

$$||y_{n-1}^* - y_n^*|| \leq \Big(\frac{\theta_{n-1} - \theta_n}{\theta_n}\Big)^{\frac{1}{2}} C_3, \text{ and} \tag{2.22}$$

$$\|J_*y_{n-1}^* - J_*y_n^*\| \leq \left(\frac{\theta_{n-1} - \theta_n}{\theta_n}\right)^{\frac{1}{p}} C_4, \tag{2.23}$$

*where $C_3$ and $C_4$ are some positive constants.*

The following important results are known.

**Lemma 2.9.** *Let $E$ be a smooth real Banach space with dual $E^*$ and the function $\phi : E \times E \to \mathbb{R}$ defined by,*

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad \textit{for } x, y \in E,$$

*where $J$ is the normalized duality mapping from $E$ into $2^{E^*}$. Then,*

$$\phi(y, x) = \phi(x, y) + 2\langle x, Jy\rangle - 2\langle y, Jx\rangle. \tag{2.24}$$

**Lemma 2.10.** *Let $X$, $X^*$ be uniformly convex and uniformly smooth real Banach spaces. Let $E = X \times X^*$ with the norm $\|z\|_E = (\|u\|_X + \|v\|_{X^*})^{\frac{1}{2}}$, for any $z = [u, v] \in E$. Let $E^* = X^* \times X$ denote the dual space of $E$. For arbitrary $x = [x_1, x_2] \in E$, define the map $J_E : E \to E^*$ by*

$$J_E(x) = J_E[x_1, x_2] := [J_X(x_1), J_{X^*}(x_2)],$$

*so that for arbitrary $z_1 = [u_1, v_1]$, $z_2 = [u_2, v_2]$ in $E$, the duality pairing $\langle \cdot, \cdot\rangle$ is given by*

$$\langle z_1, J_E\rangle := \langle u_1, J_X(u_2)\rangle + \langle v_1, J_{X^*}(v_2)\rangle.$$

*Then, $E$ is uniformly smooth and uniformly convex.*

**Lemma 2.11.** *Let $E$ be a uniformly convex and uniformly smooth real Banach and $F : E \to E^*$, $K : E^* \to E$ be maximal monotone. Define $A : E \times E^* \to E^* \times E$ by*

$$A[u, v] = [Fu - v, Kv + u] \ \forall \ [u, v] \in E \times E^*.$$

*Then, $A$ is maximal monotone.*

*Remark* 2.2. From Lemma 2.4, setting $\lambda_n := \frac{1}{\theta_n}$ where $\theta_n \to 0$ as $n \to \infty$, $z = [z_1, z_2] = J_{E\times E^*}[u, v]$ for some $[u, v] \in E \times E^*$, and $[y_n, y_n^*] := \left(J_{E\times E^*} + \frac{1}{\theta_n} A\right)^{-1} [z_1, z_2]$, we obtain that:

$$Jy_n + \frac{1}{\theta_n}(Fy_n - y_n^*) = z_1, \ \forall n \geq 0, \ \textit{and} \tag{2.25}$$

$$J_* y_n^* + \frac{1}{\theta_n}(Ky_n^* + y_n) = z_2 \ \forall n \geq 0; \tag{2.26}$$

*Remark* 2.3. Let $y_n \to y$ and $y_n^* \to y^*$. From lemma 2.4 we have that $[y_n, y_n^*]$ converges to a point in $A^{-1}0$. This implies that $[y, y^*] \in A^{-1}0$. Consequently, $A[y, y^*] = 0$, that is, $Fy - y^* = 0$ and $Ky^* + y = 0$. Hence, $y^* = Fy$ and $y + KFy = 0$.

# 3 Main Results

In theorem 3.1 below, the sequences $\{\lambda_n\}_{n=1}^{\infty}$ *and* $\{\theta_n\}_{n=1}^{\infty}$ *are in* $(0, 1)$ and are assumed to satisfy the following conditions:

*(i)* $\lambda_n, \theta_n \to 0$ as $n \to \infty$, $\sum_{n=1}^{\infty} \lambda_n\theta_n = \infty$;

*(ii)* $(\lambda_n^{\frac{1}{p}} + \lambda_n^{\frac{1}{2}}) \leq \gamma_0\theta_n$ where $\gamma_0 > 0$;

*(iii)* For $p \geq 2$, $\sum_{n=1}^{\infty} \lambda_n^{\frac{1}{p}} < \infty$, $\frac{\left(\frac{\theta_{n-1} - \theta_n}{\theta_n}\right)^{\frac{1}{p}}}{\lambda_n\theta_n} \to 0$ as $n \to \infty$.

**Theorem 3.1.** *Let $E = l_p,\ 2 \leq p < \infty,\ E^* = l_q,\ 1 < q \leq 2$ and $F : E \to E^*$, $K : E^* \to E$ be maximal monotone and bounded maps. For $u_1 \in E\ \ , v_1 \in E^*$, define the sequences $\{u_n\}$ and $\{v_n\}$ in $E$ and $E^*$, respectively by*

$$u_{n+1} = J^{-1}(Ju_n - \lambda_n(Fu_n - v_n) - \lambda_n\theta_n(Ju_n - Ju_1)),\ n \geq 1, \tag{3.1}$$

$$v_{n+1} = J_*^{-1}(J_*v_n - \lambda_n(Kv_n + u_n) - \lambda_n\theta_n(J_*v_n - J_*v_1)),\ n \geq 1, \tag{3.2}$$

*Assume that the equation $u + KFu = 0$ has a solution. Then, the sequences $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ converge strongly to $u^*$ and $v^*$, respectively, where $u^*$ is the solution of $u+KFu = 0$ with $v^* = Fu^*$.*

*Proof.* We first prove that the sequences $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ are bounded.

For $(u_n, v_n)$, $(u^*, v^*) \in E \times E^*$ where $u^*$ is a solution of (1.5) with $v^* = Fu^*$, set $w_n = (u_n, v_n)$ and $w^* = (u^*, v^*)$. Define $\Lambda : (E \times E^*) \times (E \times E^*) \to \mathbb{R}$ by

$$\Lambda(w_1, w_2) = \phi(u_1, u_2) + \phi(v_1, v_2), \tag{3.3}$$

where $w_1 = (u_1, v_1)$ and $w_2 = (u_2, v_2)$. Let $E \times E^*$ be endowed with the norm $\|(u, v)\| = (\|u\|_E^2 + \|v\|_{E^*}^2)^{\frac{1}{2}}$. We show that $\Lambda(w^*, w_n) \leq r$, for all $n \geq 1$ and for some $r > 0$. Using the fact that $F$ and $K$ are bounded, define

$$M_1 := \sup\{||(Fu - v) + \theta(Ju - Ju_1)|| : (u, v) \in B_{E\times E^*}, \theta \in (0, 1)\} + 1;$$

$$M_2 := \sup\{||(Kv + u) + \theta(J_*v - J_*v_1)|| : (u, v) \in B_{E\times E^*}, \theta \in (0, 1)\} + 1;$$
$$M_3 := \sup\{||(Ju - Ju_1|| : ||u|| \leq r_0\} + 1,\ for\ some\ r_0 > 0;$$
$$M_4 := \sup\{||J^{-1}(Ju - \lambda(Fu - v) - \lambda\theta(Ju - Ju_1)) - u|| : (u, v) \in B_{E\times E^*}, \lambda, \theta \in (0, 1)\} + 1;$$
$$M_5 := \sup\{||(Jv - Jv_1|| : ||v|| \leq r_0^*\} + 1,\ for\ some\ r_0^* > 0;$$
$$M_6 := \sup\{||J_*^{-1}(J_*v - \lambda(Kv + u) - \lambda\theta(J_*v - J_*v_1)) - v|| : (u, v) \in B_{E\times E^*}, \lambda, \theta \in (0, 1)\} + 1;$$
$$M_1^* := (M_1M_42^{p+1}pLc_2^p)^{\frac{1}{p}}$$
$$M_2^* := \Big(\frac{32M_2M_6Lc_2^2}{q - 1}\Big)^{\frac{1}{2}}$$
$$M^* =: \max\{2M_1^*M_1 + 2M_1^*M_3, 2M_2^*M_2 + 2M_2^*M_5, M_1M_4 + M_2M_6\}$$

where $c_2$ *and* $L$ are constants appearing in Lemma 2.5 and $B_{E\times E^*} = \{w \in E \times E^* : \Lambda(w^*, w) \leq r\}$. Let $r > 0$ be such that

$$\frac{r}{4} \geq \Lambda(w^*, w_1).$$

Define

$$\gamma_0 := \min\Big\{1, \frac{r}{4M^*}\Big\}.$$

Claim: $\Lambda(w^*, w_n) \leq r,\ \forall\ n \geq 1.$
The proof of this claim is by induction. By construction, we have $\Lambda(w^*, w_1) \leq r$.

Assume that $\Lambda(w^*, w_n) \leq r$ for some $n \geq 1$. This implies that

$$\phi(u^*, u_n) + \phi(v^*, v_n) \leq r,\ for\ some\ n \geq 1.$$

We prove that $\Lambda(w^*, w_{n+1}) \leq r$. Suppose, for contradiction, that this is not the case, then $\Lambda(w^*, w_{n+1}) > r$. From lemma (2.5), we have that

$$\begin{aligned} ||u_{n+1} - u_n||^p &\leq ||Ju_{n+1} - Ju_n||||u_{n+1} - u_n||2^{p+1}pLc_2^p \\ &\leq \lambda_nM_1||u_{n+1} - u_n||2^{p+1}pLc_2^p \\ &\leq \lambda_nM_1M_42^{p+1}pLc_2^p. \end{aligned}$$

This yields

$$||u_{n+1} - u_n|| \leq \lambda_n^{\frac{1}{p}} M_1^*. \tag{3.4}$$

Similarly, we obtain

$$||v_{n+1} - v_n|| \leq \lambda_n^{\frac{1}{2}} M_2^*. \tag{3.5}$$

Using the definition of $u_{n+1}$, equation (2.7) and inequality (2.8) with

$$y^* = \lambda_n(Fu_n - v_n) + \lambda_n\theta_n(Ju_n - Ju_1),$$

we obtain

$$\begin{aligned}
\phi(u^*, u_{n+1}) &= V(u^*, Ju_n - \lambda_n(Fu_n - v_n) - \lambda_n\theta_n(Ju_n - Ju_1)) \\
&\leq V(u^*, Ju_n) - 2\Big\langle u_{n+1} - u^*, \lambda_n(Fu_n - v_n) + \lambda_n\theta_n(Ju_n - Ju_1)\Big\rangle \\
&= \phi(u^*, u_n) - 2\Big\langle u_{n+1} - u_n, \lambda_n\Big((Fu_n - v_n) + \theta_n(Ju_n - Ju_1)\Big)\Big\rangle \\
&\quad -2\Big\langle u_n - u^*, \lambda_n\Big((Fu_n - v_n) + \theta_n(Ju_n - Ju_1)\Big)\Big\rangle
\end{aligned}$$

Which implies that

$$\begin{aligned}
\phi(u^*, u_{n+1}) &\leq \phi(u^*, u_n) + 2\|u_{n+1} - u_n\|\lambda_n M_1 \\
&\quad -2\lambda_n\Big\langle u_n - u^*, (Fu_n - v_n) + \theta_n(Ju_n - Ju_1)\Big\rangle
\end{aligned} \tag{3.6}$$

Observe that using the monotonicity of $F$ and $J$, we have:

$$\begin{aligned}
&\Big\langle u_n - u^*, (Fu_n - v_n) + \theta_n(Ju_n - Ju_1)\Big\rangle \\
\geq\ &\langle u_n - u^*, (Fu^* - v_n)\rangle + \theta_n\langle u_n - u_{n+1}, Ju_n - Ju_{n+1}\rangle \\
&+\theta_n\langle u_n - u_{n+1}, Ju_{n+1} - Ju_1\rangle + \theta_n\langle u_{n+1} - u^*, Ju_n - Ju_{n+1}\rangle \\
&+\theta_n\langle u_{n+1} - u^*, Ju_{n+1} - Ju_1\rangle \\
\geq\ &\langle u_n - u^*, (Fu^* - v_n)\rangle - \theta_n||u_n - u_{n+1}||||Ju_{n+1} - Ju_1|| \\
&-\theta_n||u_{n+1} - u^*||||Ju_n - Ju_{n+1}|| + \theta_n\langle u_{n+1} - u^*, Ju_{n+1} - Ju_1\rangle.
\end{aligned}$$

Substituting into inequality (3.6), we obtain

$$\begin{aligned}
\phi(u^*, u_{n+1}) &\leq \phi(u^*, u_n) + 2\|u_{n+1} - u_n\|\Big|\Big|\lambda_n\Big((Fu_n - v_n) + \theta_n(Ju_n - Ju_1)\Big)\Big|\Big| \\
&\quad -2\lambda_n\langle u_n - u^*, (Fu^* - v_n)\rangle + 2\lambda_n\theta_n||u_n - u_{n+1}||||Ju_{n+1} - Ju_1|| \\
&\quad +2\lambda_n\theta_n||u_{n+1} - u^*||||Ju_n - Ju_{n+1}|| - 2\lambda_n\theta_n\langle u_{n+1} - u^*, Ju_{n+1} - Ju_1\rangle.
\end{aligned}$$

Now, using inequality (2.16) of lemma 2.7 and inequality (3.4), we have that

$$\begin{aligned}
\phi(u^*, u_{n+1}) &\leq \phi(u^*, u_n) - \lambda_n\theta_n\phi(u^*, u_{n+1}) + \lambda_n\theta_n\phi(u^*, u_1) + \lambda_n(\lambda_n^{\frac{1}{p}} M_1^*)(2M_1) + 2\lambda_n\theta_n(\lambda_n M_1)M_4 \\
&\quad +\lambda_n\theta_n(\lambda_n^{\frac{1}{p}} M_1^*)(2M_3) - 2\lambda_n\langle u_n - u^*, (Fu^* - v_n)\rangle \\
&\leq \phi(u^*, u_n) - \lambda_n\theta_n\phi(u^*, u_{n+1}) + \lambda_n\theta_n\phi(u^*, u_1) + \lambda_n[\lambda_n^{\frac{1}{p}}(2M_1^*M_1 + 2M_1^*M_3)] \qquad (3.7) \\
&\quad +2\lambda_n\theta_n(\lambda_n M_1)M_4 - 2\lambda_n\langle u_n - u^*, (Fu^* - v_n)\rangle
\end{aligned}$$

Similarly, using the fact that $K$ and $J$ are monotone, inequality (2.16) of lemma 2.7 and inequality (3.5), we have

$$\begin{aligned}\phi(v^*, v_{n+1}) &\leq \phi(v^*, v_n) - \lambda_n\theta_n\phi(v^*, v_{n+1}) + \lambda_n\theta_n\phi(v^*, v_1) + \lambda_n[\lambda_n^{\frac{1}{2}}(2M_2^*M_2 + 2M_2^*M_5)] \\ &\quad +2\lambda_n\theta_n(\lambda_n M_2)M_6 - 2\lambda_n\langle v_n - v^*, (Kv^* + u_n)\rangle\end{aligned}$$

Observe that since $u^* + KFu^* = 0$, setting $Fu^* = v^*$, we obtain that $Kv^* = -u^*$, and these equations yield

$$2\lambda_n\langle u_n - u^*, (v_n - Fu^*)\rangle + 2\lambda_n\langle v_n - v^*, -(Kv^* + u_n)\rangle = 0,$$

so that adding (3.7) and (3.8), we obtain

$$r < \Lambda(w^*, w_{n+1}) \leq \Lambda(w^*, w_n) - \lambda_n\theta_n\Lambda(w^*, w_{n+1}) + \lambda_n\theta_n\Lambda(w^*, w_1) + \lambda_n(\lambda_n^{\frac{1}{p}} + \lambda_n^{\frac{1}{2}})M^* + \lambda_n\theta_n(2\lambda_n M^*)$$

So we have

$$\begin{aligned}r < \Lambda(w^*, w_{n+1}) &\leq \Lambda(w^*, w_n) - \lambda_n\theta_n\Lambda(w^*, w_{n+1}) + \lambda_n\theta_n\Lambda(w^*, w_1) + \lambda_n\theta_n\gamma_0 M^* + \lambda_n\theta_n\gamma_0 M^* \\ &\leq r - \lambda_n\theta_n r + \lambda_n\theta_n\frac{r}{4} + \lambda_n\theta_n\frac{r}{4} + \lambda_n\theta_n\frac{r}{4} < r.\end{aligned}$$

This is a contradiction, hence, $\Lambda(w^*, w_{n+1}) \leq r$ and so $\Lambda(w^*, w_n) \leq r$ for all $n \geq 1$. Consequently, we have $\phi(u^*, u_n) \leq r$ and $\phi(v^*, v_n) \leq r$ for all $n \geq 1$. Thus from inequality (2.5), we have that $\{u_n\}_{n\geq 1}$ and $\{v_n\}_{n\geq 1}$ are bounded.

We now prove that $\{u_n\}$ converges strongly to a solution of the Hammerstein equation.

Using equation (2.7), lemmas 2.9 and 2.1, with $y^* = \lambda_n(Fu_n - v_n) + \lambda_n\theta_n(Ju_n - Ju_1)$, we have

$$\begin{aligned}\phi(y_n, u_{n+1}) &= \phi(y_n, J^{-1}(Ju_n - \lambda_n(Fu_n - v_n) - \lambda_n\theta_n(Ju_n - Ju_1))) \\ &\leq V(y_n, Ju_n) - 2\langle u_{n+1} - y_n, \lambda_n(Fu_n - v_n) + \lambda_n\theta_n(Ju_n - Ju_1)\rangle \\ &= \phi(u_n, y_n) + 2\langle u_n, Jy_n\rangle - 2\langle y_n, Ju_n\rangle - 2\lambda_n\langle u_{n+1} - y_n, (Fu_n - v_n) \\ &\quad +\theta_n(Ju_n - Ju_1)\rangle \\ &= V(u_n, Jy_n) + 2\langle u_n, Jy_n\rangle - 2\langle y_n, Ju_n\rangle - 2\lambda_n\langle u_{n+1} - y_n, (Fu_n - v_n) \\ &\quad +\theta_n(Ju_n - Ju_1)\rangle \\ &\leq V(u_n, Jy_{n-1}) - 2\langle y_n - u_n, Jy_{n-1} - Jy_n\rangle + 2\langle u_n, Jy_n)\rangle - 2\langle y_n, Ju_n)\rangle \\ &\quad -2\lambda_n\langle u_{n+1} - y_n, (Fu_n - v_n) + \theta_n(Ju_n - Ju_1)\rangle \\ &= \phi(y_{n-1}, u_n) + 2\langle y_{n-1}, Ju_n\rangle - 2\langle u_n, Jy_{n-1}\rangle - 2\langle y_n - u_n, Jy_{n-1} - Jy_n\rangle \\ &\quad +2\langle u_n, Jy_n\rangle - 2\langle y_n, Ju_n\rangle - 2\lambda_n\langle u_{n+1} - y_n, (Fu_n - v_n) + \theta_n(Ju_n - Ju_1)\rangle \\ &= \phi(y_{n-1}, u_n) + 2\langle y_{n-1} - y_n, Ju_n\rangle + 2\langle y_n, Jy_n - Jy_{n-1}\rangle \\ &\quad -2\lambda_n\langle u_{n+1} - y_n, (Fu_n - v_n) + \theta_n(Ju_n - Ju_1)\rangle\end{aligned}$$

Appying monotonicity of $F$ and using equations (2.25), (2.17), (3.4), (2.20) and (2.21), we have

$$\begin{aligned}
\phi(y_n, u_{n+1}) &\leq \phi(y_{n-1}, u_n) + ||y_n - y_{n-1}||K_1 + ||Jy_n - Jy_{n-1}||K_2 + 2\lambda_n||u_{n+1} - u_n||M_1 \\
&\quad -2\lambda_n\langle u_n - y_n, (Fu_n - v_n) + \theta_n(Ju_n - Ju_1)\rangle \\
&= \phi(y_{n-1}, u_n) + ||y_n - y_{n-1}||K_1 + ||Jy_n - Jy_{n-1}||K_2 + 2\lambda_n||u_{n+1} - u_n||M_1 \\
&\quad -2\lambda_n\langle u_n - y_n, (Fu_n - v_n) + \theta_n(Ju_n - Jy_n - \frac{1}{\theta_n}(Fy_n - y_n^*))\rangle \\
&\leq \phi(y_{n-1}, u_n) + ||y_n - y_{n-1}||K_1 + ||Jy_n - Jy_{n-1}||K_2 + 2\lambda_n||u_{n+1} - u_n||M_1 \\
&\quad -2\lambda_n\langle u_n - y_n, y_n^* - v_n\rangle - 2\lambda_n\theta_n\langle u_n - y_{n-1}, Ju_n - Jy_{n-1}\rangle \\
&\quad -2\lambda_n\theta_n\langle u_n - y_{n-1}, Jy_{n-1} - Jy_n\rangle - 2\lambda_n\theta_n\langle y_{n-1} - y_n, Ju_n - Jy_n\rangle \\
&\leq \phi(y_{n-1}, u_n) + ||y_n - y_{n-1}||K_1 + ||Jy_n - Jy_{n-1}||K_2 + 2\lambda_n||u_{n+1} - u_n||M_1 \\
&\quad -\lambda_n\theta_n\phi(y_{n-1}, u_n) + ||Jy_n - Jy_{n-1}||K_3 + ||y_n - y_{n-1}||K_4 - 2\lambda_n\langle u_n - y_n, y_n^* - v_n\rangle \\
&\leq \phi(y_{n-1}, u_n) - \lambda_n\theta_n\phi(y_{n-1}, u_n) + \Big(\frac{\theta_{n-1} - \theta_n}{\theta_n}\Big)^{\frac{1}{p}} C_1K_5 \qquad (3.8) \\
&\quad +\Big(\frac{\theta_{n-1} - \theta_n}{\theta_n}\Big)^{\frac{1}{2}} C_2K_6 + 2\lambda_n(\lambda_n^{\frac{1}{p}} M_1^*)M_1 - 2\lambda_n\langle u_n - y_n, y_n^* - v_n\rangle;
\end{aligned}$$

where $K_1,\ K_2,\ K_3,\ K_4$ are positive constants and $K_5 = K_1 + K_4,\ K_6 = K_2 + K_3$.

Similarly, applying monotonicity of $K$ and using equations (2.26), (2.17), (3.5), (2.22) and (2.23), we have

$$\begin{aligned}
\phi(y_n^*, v_{n+1}) &\leq \phi(y_{n-1}^*, v_n) - \lambda_n\theta_n\phi(y_{n-1}^*, v_n) + \Big(\frac{\theta_{n-1} - \theta_n}{\theta_n}\Big)^{\frac{1}{2}} C_3K_5^* \qquad (3.9) \\
&\quad +\Big(\frac{\theta_{n-1} - \theta_n}{\theta_n}\Big)^{\frac{1}{p}} C_4K_6^* + 2\lambda_n(\lambda_n^{\frac{1}{p}} M_2^*)M_2 - 2\lambda_n\langle v_n - y_n^*, u_n - y_n\rangle;
\end{aligned}$$

where $C_3,\ C_4,\ K_5^*$ and $K_6^*$ are positive constants.

Hence, adding equations (3.8) and (3.9) we have

$$\begin{aligned}
\Lambda(p_n, w_{n+1}) &\leq \Lambda(p_{n-1}, w_n) - \lambda_n\theta_n\Big(\phi(y_{n-1}, u_n) + \phi(y_{n-1}^*, v_n)\Big) + 2\lambda_n(\lambda_n)^{\frac{1}{p}} M_1^*M_1 \\
&\quad +2\lambda_n(\lambda_n)^{\frac{1}{2}} M_2^*M_2 + \Big(\frac{\theta_{n-1} - \theta_n}{\theta_n}\Big)^{\frac{1}{p}} C_1K_5 + \Big(\frac{\theta_{n-1} - \theta_n}{\theta_n}\Big)^{\frac{1}{p}} C_4K_6^* \\
&\quad +\Big(\frac{\theta_{n-1} - \theta_n}{\theta_n}\Big)^{\frac{1}{2}} C_2K_6 + \Big(\frac{\theta_{n-1} - \theta_n}{\theta_n}\Big)^{\frac{1}{2}} C_3K_5^*.
\end{aligned}$$

Letting $M^* = \max\{C_1K_5 + C_4K_6^*, C_2K_6 + C_3K_5^*, 2M_1^*M_1, 2M_2^*M_2\}$, we have

$$\begin{aligned}
\Lambda(p_n, w_{n+1}) &\leq \Lambda(p_{n-1}, w_n) - 2\lambda_n\theta_n\Lambda(p_{n-1}, w_n) + \lambda_n^{\frac{1}{p}} M^* + \lambda_n^{\frac{1}{2}} M^* \\
&\quad +\Big(\frac{\theta_{n-1} - \theta_n}{\theta_n}\Big)^{\frac{1}{p}} M^* + \Big(\frac{\theta_{n-1} - \theta_n}{\theta_n}\Big)^{\frac{1}{2}} M^*.
\end{aligned}$$

Setting

$\rho_n := \Lambda(p_{n-1}, w_n)$; $\beta_n := \lambda_n\theta_n$; $\zeta_n := \left(\frac{\left(\frac{\theta_{n-1}-\theta_n}{\theta_n}\right)^{\frac{1}{p}} M^*}{\lambda_n\theta_n} + \frac{\left(\frac{\theta_{n-1}-\theta_n}{\theta_n}\right)^{\frac{1}{2}} M^*}{\lambda_n\theta_n}\right)$; $\gamma_n := \lambda_n^{\frac{1}{p}} M^* + \lambda_n^{\frac{1}{2}} M^*$;

we have

$$\rho_{n+1} \leq (1 - \beta_n)\rho_n + \beta_n\zeta_n + \gamma_n,\ n \geq 1.$$

It now follows from Lemma (2.4) that $\rho_n \to 0$ as n$\to \infty$, *i.e.*, $\Lambda(p_{n-1}, w_n) \to 0$ as n$\to \infty$. Consequently, by lemma (2.3), we obtain that $\lim||u_n - y_{n-1}|| = 0$. Hence using remark 2.3, we have that the sequence $\{u_n\}_{n=1}^{\infty}$ converges strongly to a solution of (1.5). □

**Example 3.2.** *Real sequences that satisfy the conditions $(i)-(iii)$ of theorem 3.1 are the following:*

$$\lambda_n = (n+1)^{-a} \text{ and } \theta_n = (n+1)^{-b}, \quad n \geq 1.$$
$$0 < b < \frac{a}{p}, \quad \text{and } a + b < \frac{1}{p}.$$

*For example, take $a := \frac{1}{(p+1)}$ and $b := \frac{1}{2p(p+1)}$ then conditions $(i)$ - $(iii)$ are satisfied.*

*Remark* 3.1. Theorem 3.1 is an extension of theorem 1.1 to $l_p$ spaces $(2 \leq p < \infty)$.

**Open question 1.** *Does Theorem 3.1 hold in $l_p$ spaces for all $p$ such that $1 < p < 2$?*

*Remark* 3.2. (see e.g., Alber [14], p.36) The analytical representations of duality mappings are known in a number of Banach spaces. For instance, in the spaces $l^p$, $L^p(G)$ and $W_m^p(G)$, $p \in (1, \infty)$, $p^{-1} + q^{-1} = 1$, respectively,

$$Jx = ||x||_{l^p}^{2-p} y \in l^q, \quad y = (|x_1|^{p-2}x_1, |x_2|^{p-2}x_2, ...), \quad x = (x_1, x_2, ...),$$
$$Jx = ||x||_{L^p}^{2-p} |x(s)|^{p-2}x(s) \in L^q(G), \quad s \in G,$$

and

$$Jx = ||x||_{W_m^p}^{2-p} \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^{\alpha}(|D^{\alpha}x(s)|^{p-2}D^{\alpha}x(s)) \in W_{-m}^q(G), m > 0, s \in G$$

# 4 Conclusion

Theorem 3.1 is a strong convergence theorem

which extends Theorem 1.1 to a space more general than Hilbert space.

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# Competing Interests

The authors have declare that no competing interests exist.

# References


[1] Pascali D, Sburian S. Nonlinear mappings of monotone type. Editura Academia Bucuresti, Romania; 1978.

[2] Dolezale V. Monotone operators and its applications in automation and network theory. Studies in Automation and Control, Vol. 3, Elsevier Science publ., New York; 1979.

[3] Brézis H, Browder FE. Some new results about Hammerstein equations. Bull. Amer. Math. Soc. 1974;80:567-572.

[4] Brézis H, Browder FE. Existence theorems for nonlinear integral equations of Hammerstein type. Bull. Amer. Math. Soc. 1975;81:73-78.

[5] Brézis H, Browder FE. Nonlinear integral equations and systems of Hammerstein type. Bull. Amer. Math. Soc. 1976;82:115-147.

[6] Browder FE. Nonlinera Functional Analysis and Nonlinear Integral equations of Hammerstein and Uryshon type. Contributions to Nonlinear Functional Analysis, Academic Press. 1971:425-500.

[7] Browder FE, De Figueiredo DG, Gupta CP. Maximal mono- tone operators and nonlinear integral equations of Hammerstein type. Bull. Amer. Math. Soc. 1970;76:700-705.

[8] Mann WR. Mean value methods in iteration. Proc. Amer. Math. Soc. 1953;4:506-510.

[9] Chidume CE, Osilike MO. Iterative solutions of nonlinear integral equations of Hammerstein-type. J. Nigerian Math. Soc. 1992;11:9-18 (MR96c:65207).

[10] Chidume CE, Zegeye H. Iterative approximation of solutions of nonlinear equation of Hammerstein-type. Abstr. Appl. Anal. 2003;6:353-367.

[11] Chidume CE, Zegeye H. Approximation os solutions of nonlinear equations of monotone and Hammerstein-type. Appl. Anal. 2003;82(8):747-758.

[12] Chidume CE, Djitte N. Approximation of Solutions of Nonlinear Integral Equations of Hammerstein Type. ISRN Mathematical Analysis. vol. 2012, Article ID 169751. 2012;12.

[13] Chidume CE. Geometric Properties of Banach Spaces and Nonlinear iterations. vol. 1965 of Lectures Notes in Mathematics. Springer, London, UK; 2009.

[14] Alber Ya, Ryazantseva I. Nonlinear Ill Posed Problems of Monotone Type. Springer, London, UK, 2006. J. Nonlinear Convex Anal. 2014;15(5):851-865.

[15] Alber Ya. Metric and generalized projection operators in Banach spaces: Properties and applications. In Theory and Applications of Nonlinear Operators of Accretive and Monotone Type (A. G. Kartsatos, Ed.), Marcel Dekker, New York. 1996;15-50.

[16] Alber Ya, Guerre-Delabriere S. On the projection methods for fixed point problems. Analysis (Munich). 2001;21(1):17-39.

[17] Kamimura S, Takahashi W. Strong convergence of a proximal-type algorithm in a Banach space. SIAMJ. Optim. 2002;13(3):938-945.

[18] Reich S. A weak convergence theorem for the alternating methods with Bergman distance. in: A. G. Kartsatos (Ed.), Theory and Applications of Nonlinear Operators of Accretive and Monotone Type, in Lecture notes in pure and Appl. Math., Dekker, New York. 1996;178:313-318.

[19] Xu HK. Iterative algorithms for nonlinear operators. J. London Math. Soc. 2002;66(2):240-256.

[20] Reich S. Constructive techniques for accretive and monotone operators. Applied non-linear analysis, Academic Press, New York. 1979;335-345.

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