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Piece–wise Information for the Truncated Cosinor Model

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Abstract

The target of this paper is to examine the average–per–observation information matrix for the truncated cosinor model. We state and prove an orthogonal decomposition of this matrix so that the total information can be obtained as a result of three particular parts. The total information is presented piece–wise in three components. Each component is easily represented. Therefore the total information can be checked through three different points on the collection of the information.

Keywords: Optimal design theory, Design measure, D–optimality, c–optimality, cosinor model, information matrix

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1 Introduction

In most of the real life cases we face the situation where neither the models are perfect indeed, nor the data. In such cases uncertainty is the permanent escort of knowledge and we try to reduce it. This knowledge is relevant with the obtained information from the collected data set. Therefore, if we "break" the information into "parts", we can particular investigate the provided total information. So we try to obtain a piece–wise information while the sum of the total information we are referred to is the Fisher's information. Moreover, this is crucial as a representation of the uncertainty formalizes incomplete information and vice versa. As the reduction of uncertainty requires the identification of the sources, as well as the elaboration of the appropriate methods to qualify, it is worth it to break the total estimated information into parts. We are using three different parts of information to sum up to the total. The nonlinear model we discuss in this paper reflects data either from biological rhythms, [1] or from engineering situations, [2]. The adopted information measure is the Fisher's (parametric) information measure.

Consider the regression model linking the response y with the deterministic portion $f(u;\vartheta)$ and the stochastic portion e known as error, of the form

$$y(t) = f(u;\vartheta) + e,$$

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with ϑ being the vector of parameters $\vartheta \in \Theta \subseteq \mathbb{R}^p$ and u the input variable. We let $\eta = \mathrm{E}(y|u) = f(u;\vartheta)$ and assume that the error e comes from a distribution with zero mean and variance σ^2. When inference is required this distribution is asked to be the normal. Then Fisher's information matrix is defined to be

$$\mathrm{I}(u;\vartheta) = \sigma^{-2}(\nabla\eta)(\nabla\eta)^{\mathrm{T}},$$

where $\nabla\eta$ is the usual gradient of f. Then, for the n–point design measure, the average-per-observation (a–p–o) information matrix $\mathbf{M}$ defined, for the discrete case, to be, [3].

$$\mathbf{M}(\vartheta,\xi) = \tfrac{1}{n}\sum_{i=1}^{n}\mathrm{I}(u_i;\vartheta).$$

The defined matrix $\mathbf{M}(\vartheta,\xi)$ for this particular "cosinor model" is the one which we try to obtain piece–wise as a summation of different a–p–o information matrices. In principle, the linear regression case is distinguished from the non-linear by the fact that the a–p–o information matrix does not depend on the parameter ϑ, as in the non-linear case we investigate.

2 Background

Some diural and engineering rhythms can be described, with the known as the cosinor model,

$$y(t) = f(t;\vartheta) + e \ \text{ with } \ \eta = \mathrm{E}(y|u) = f(u;\vartheta) = \vartheta_0 + \vartheta_1\cos(\omega t + \vartheta_2), \tag{2.1}$$

where $y(t)$ is the response at time t, i.e. the variable of the rhythm we study, ϑ_0 is the *mesor*, i.e. the "mean" value about which oscillation occurs, ϑ_1 is the *amplitude*, i.e. the half difference between the highest and lowest value during the oscillation in a complete cycle (360° or 24 hours), ϑ_2 is the *acrophase*, i.e. timing of high point in degrees, ω is angular frequency, i.e. degrees/unit time ($2\pi = 360^\circ$ corresponds to a complete cycle, and e is the error term under the usual normal assumption. For an application in biorhythms fitting the model (2.1), see [4] and [5].

From an applications point of view (clinical or engineering) the ratio $g = g(\vartheta_0,\vartheta_1) = \vartheta_1/\vartheta_0$ is the parameter of interest. This represents the ratio of the amplitude of the cyclic variation to the overall mean.

Expanding the cosine term in (2.1) we have

$$\eta(x;\vartheta) = \vartheta_0 x_0 + \beta_1 x_1 + \beta_2 x_2, \tag{2.2}$$

where $\beta_1 = \vartheta_1\cos\vartheta_2$, $\beta_2 = -\vartheta_1\sin\vartheta_2$ with $x_0 = 1$, $x_1 = \cos 2\pi t$ and $x_2 = \sin 2\pi t$. Therefore, model (2.1) can be written as

$$y(t) = W^{\mathrm{T}}(t)\beta + e, \quad \beta = (\vartheta_0,\beta_1,\beta_2), \ \ W^{\mathrm{T}}(t) = (x_0,x_1,x_2). \tag{2.3}$$

When the model (2.3) is fitted, estimates for ϑ_1 and ϑ_2 can be obtained, due to the new parameterization, through the relations

$$\hat{\vartheta}_1 = \sqrt{\hat{\beta}_1^2 + \hat{\beta}_2^2} \ \text{ and } \ \hat{\vartheta}_2 = \hat{\omega} + \kappa \tag{2.4}$$

where $\hat{\omega} = \arctan|\hat{\beta}_2/\hat{\beta}_1|$, κ is an appropriate constant. and the estimates of β are

2.1 D–optimal design

The target for the model (2.2), is to estimate the non-linear ratio g

$$g = g(\vartheta_0,\vartheta_1) = \vartheta_1/\vartheta_0, \tag{2.5}$$

as good as possible. Therefore, we consider optimum experimental designs for the estimation. We shall focus on the case $0 < g < 1$. The case $g < 0$ have no interest in applications, see **?** for details, while the case $g > 1$ provide the dual problem as far as the optimal design points and the corresponding optimal design measure are concerned. For the model (2.5) the design space, say X, is a circle, defined by

$$x_0 = 1, \quad x_1^2 + x_2^2 = 1.$$

The center of the circle lies on the x_0 axis is at point $(1, 0)$. It follows then, [6] (p. 75), that the points of the D-optimal design must lie on the given circle.

Interest is focused on the estimation of the ratio (2.5) written, due to (2.4), as

$$g = \tfrac{1}{\vartheta_0}\sqrt{\beta_1^2 + \beta_2^2}. \tag{2.6}$$

Thus the approximate variance of g is

$$\mathrm{Var}(\hat{g}) =\cong \tfrac{1}{n}(\nabla g)^{\mathrm{T}}\mathbf{M}^{-1}(\nabla g), \tag{2.7}$$

where ∇g, after some calculations, equals to

$$(\nabla g)^{\mathrm{T}} = \left(-\tfrac{1}{\vartheta_0^2}\sqrt{\beta_1^2 + \beta_2^2},\ \tfrac{\beta_1}{\vartheta_0}\sqrt{\beta_1^2 + \beta_2^2},\ \tfrac{\beta_2}{\vartheta_0}\sqrt{\beta_1^2 + \beta_2^2}\right) = \tfrac{1}{\vartheta_0}\left(-\tfrac{\vartheta_1}{\vartheta_0},\ \tfrac{\beta_1}{\vartheta_1},\ \tfrac{\beta_2}{\vartheta_1}\right). \tag{2.8}$$

For the model (2.2), the a–p–o information matrix for this four design points: t, $t + \frac{1}{4}$, $t + \frac{1}{2}$, $t + \frac{3}{4}$ is reduced to

$$n\mathbf{M} = n\,\mathrm{diag}\left(1, \tfrac{1}{2}, \tfrac{1}{2}\right).$$

Substituting (2.8) in (2.7) we obtain that the approximate variance, say $\mathrm{V}_{4\mathrm{D}}$, of an equally–spaced, equally–weighted, 4–point D–optimal design is

$$\mathrm{V}_{4\mathrm{D}} = \frac{\sigma^2}{n\vartheta_0^2}\left(\frac{\vartheta_1^2}{\vartheta_0^2} + 2\right). \tag{2.9}$$

Note that, from (2.7), the problem can be considered, approximately, equivalent to a locally c–optimal design where the ray "c" is given by $(\nabla g)^{\mathrm{T}}$. For the construction of a coptimal design see [1] and [3]. We briefly discussed it in the next Section.

2.2 c–optimal design

For given ϑ_0, ϑ_1, ϑ_2 and therefore β_1, β_2, the locally c–optimal design problem is the one satisfying

$$\min\left\{c^{\mathrm{T}}\mathbf{M}^{-1}(\xi)c\right\}, \tag{2.10}$$

with the choice of $c = (c_0, c_1, c_2)^{\mathrm{T}} = \nabla g$ as in (2.7) and

$$n\mathbf{M}(\xi) = \begin{pmatrix} n & \sum\cos 2\pi t_i & \sum\sin 2\pi t_i \\ \sum\cos 2\pi t_i & \sum\cos^2 2\pi t_i & \sum\cos 2\pi t_i \sin 2\pi t_i \\ \sum\sin 2\pi t_i & \sum\cos 2\pi t_i \sin 2\pi t_i & \sum\sin^2 2\pi t_i \end{pmatrix}, \tag{2.11}$$

when the 4–point design was considered to be t,$t+\frac{1}{4}$, $t+\frac{1}{2}$, $t+\frac{3}{4}$, equivalently in angles $2\pi t$, $2\pi t+\frac{\pi}{2}$, $2\pi t+\pi$, $\pi t+\frac{3\pi}{2}$. In principle we require an optimal measure ξ on $[0, 1)$ to solve (2.10). [7] developed a geometrical approach to finding c–optimal designs. With this in mind, and considering the reflection $-X$, of the design space X, a cylinder is formed, connecting X and $-X$ with the x_0–axis as axis and directrix of the circle X. The geometry of the problem suggests the use of Elfving′s theorem, see [7] , to tackle the two cases described above, see for details [1].

Eventually, for $\vartheta_1/\vartheta_0 < 1$, we allocate the optimum design measure, say ξ_c,

$$\xi_c \;=\; 0.5\left(1-\tfrac{\vartheta_1}{\vartheta_0}\right) \text{ at the optimal design point } -\tfrac{1}{2\pi}\vartheta_2, \tag{2.12}$$

$$1-\xi_c \;=\; 0.5\left(1+\tfrac{\vartheta_1}{\vartheta_0}\right) \text{ at the optimal design point } \pi-\tfrac{1}{2\pi}\vartheta_2. \tag{2.13}$$

Notice the essential difference for the D–optimal design measure is $\xi_{\mathrm{D}} = \frac{1}{4}$

For the two point design the corresponding 3×3 matrix $\mathbf{M} = \mathbf{M}(\vartheta,\xi)$ is singular with $\operatorname{rank}\mathbf{M} = 2$. Considering $\mathbf{M}(\vartheta,\xi)$ as in (2.11), for this particular case it is easy to verify that the matrix $\mathbf{M}(\vartheta,\xi)$, when substituting ξ_c as in (2.12), is

$$\mathbf{M}(\vartheta,\xi_c) = \begin{pmatrix} 1 & -(\vartheta_1/\vartheta_0)\cos\vartheta_2 & (\vartheta_1/\vartheta_0)\sin\vartheta_2 \\ -(\vartheta_1/\vartheta_0)\cos\vartheta_2 & \cos^2\vartheta_2 & -\cos\vartheta_2\sin\vartheta_2 \\ (\vartheta_1/\vartheta_0)\sin\vartheta_2 & -\cos\vartheta_2\sin\vartheta_2 & \sin^2\vartheta_2 \end{pmatrix}. \tag{2.14}$$

To solve (2.10), the generalized inverse $\mathbf{M}^{-}(\vartheta,\xi_c)$ is needed. Using a matrix result, **?**, with $\vartheta_2 \neq \frac{\pi}{2}$,

$$\mathbf{M}^{c}(\vartheta,\xi_c) = \begin{pmatrix} \cos^2\vartheta_2 & \frac{\vartheta_1}{\vartheta_0}\cos\vartheta_2 & 0 \\ \frac{\vartheta_1}{\vartheta_0}\cos\vartheta_2 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \left[\cos\vartheta_2\left(1-\frac{\vartheta_1^2}{\vartheta_0^2}\right)\right]^{-2}. \tag{2.15}$$

Hence, for the design measure $\xi = \xi_c$ for the c–optimal case, and

$$c^{\mathrm{T}} = (\nabla g)^{\mathrm{T}} = \tfrac{1}{\vartheta_0}\left(-\tfrac{\vartheta_1}{\vartheta_0},\ \tfrac{\beta_1}{\vartheta_1},\ \tfrac{\beta_2}{\vartheta_1}\right) = \tfrac{1}{\vartheta_0}\left(-\tfrac{\vartheta_1}{\vartheta_0},\ \cos\vartheta_2,\ -\sin\vartheta_2\right),$$

we obtain

$$c^{\mathrm{T}}\mathbf{M}^{-}(\vartheta,\xi)c = \vartheta_0^{-2}. \tag{2.16}$$

Therefore the approximate variance $\mathrm{V}_{2\mathrm{c}}$ for the 2–point c–optimal design is

$$\mathrm{V}_{2\mathrm{c}} = \mathrm{Var}\left(c^{\mathrm{T}}\hat{\vartheta}\right) = \vartheta_0^{-2}\tfrac{\sigma^2}{n}, \quad \vartheta_1/\vartheta_0 < 1. \tag{2.17}$$

Comparing (2.17) and (2.9) we can easily verify that $\mathrm{V}_{4\mathrm{D}}/\mathrm{V}_{2\mathrm{c}} < 3$.

The case $g > 1$ is the dual problem as far the design measures are concerned; see **?**, and as it has no practical application we are not referring to it.

3 Truncated Design Space

An important practical difficulty with the optimal designs in Section 2 is that they require measurements to be made when the response function is maximum and minimum. The latter typically occurs, at least in some biorhythms, in the early hours of the morning, and this might create problems when we are collecting the data set for the experiment. The situation is typical when the "blood pressure" is to be measured.

It might be desirable for the design to be restricted to more social hours, i.e. avoid taking measurements during the night. We restrict the design to a portion, say $1-T$, of the day where T is the length of the night–time period, e.g. 11pm till 7am. Moreover we assume that the minimum of the response function occurs at the middle of T and the maximum, in $1-T$, occurs at the middle of this interval. Any design depends on ϑ_1/ϑ_0 and ϑ_2. Also, the restriction on time means that the new design space, say X_N, is no longer a circle and hence the idea of a full cylinder is no any longer available. The cylinder will be "truncated". That is, the equation of the line through the points $O(0,0)$ and $R_1(-\vartheta_1/\vartheta_0, 1)$ is

$$y = (-\vartheta_1/\vartheta_0)x, \tag{3.1}$$

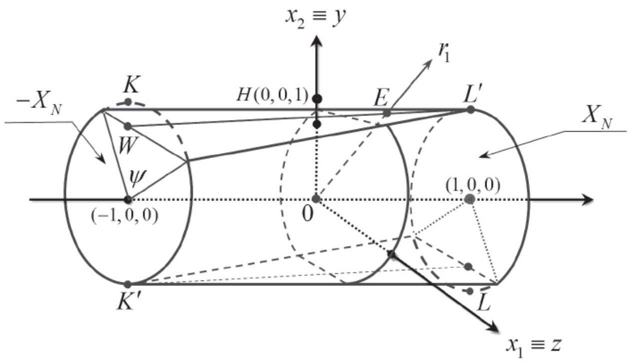

Figure 1: The truncated space X_N and its corresponded truncated cylinder.

see also Figure 1 below and Figure 2.

As T is the portion of the day excluded from the design space X, we have that $2\psi = 2\pi T$, see Figure 1. The equation of the line through the points $L'(1,1)$ and $W(-1,\cos\psi)$ with ψ, being the angle corresponding to the portion of 2π which is equivalent to $T/2$, is

$$\frac{y-1}{\cos\psi - 1} = \frac{x-1}{-1-1}.$$

Therefore, substituting $\cos\psi = \cos\pi T$, we get

$$y = 0.5(1+\cos\pi T) + 0.5(1-\cos\pi T)x, \tag{3.2}$$

or, from the simultaneous equations (3.1) and (3.2) we get

$$x = \frac{0.5(1+\cos\pi T)}{0.5\cos\pi T - 0.5 - (\vartheta_0/\vartheta_1)}. \tag{3.3}$$

Then, relation (3.3) provides x and the corresponding y measurement from (3.2). It is easy to see that $x = -\vartheta_1/\vartheta_0$ for $T = 0$, and therefore $y = 1$.

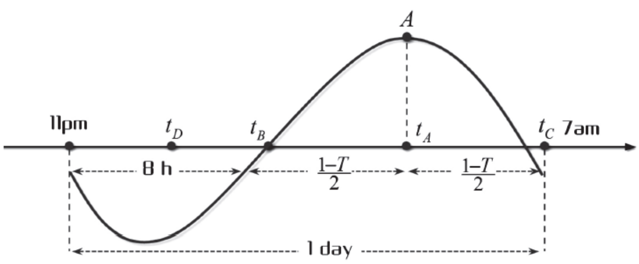

Figure 2: A typical situation when the rhythm is truncated.

Now, using Elfving's theorem (recall Figure 3) we have that

$$\frac{1-\xi'}{\xi'} = \frac{1+x}{1-x}, \quad \xi' = 0.5(1-x).$$

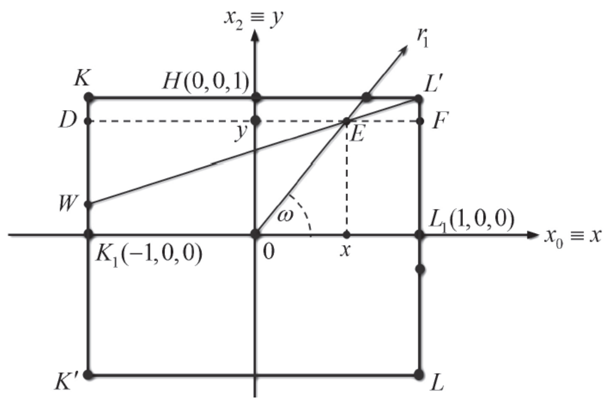

Figure 3: Side elevation of Figure 1.

There,the appropriate optimal design measure for the truncated case $\xi_{\rm t}$ is $1-\xi'$ with

$$\xi' = \frac{-\frac{1}{2}(\frac{\vartheta_0}{\vartheta_1}+1)}{\frac{1}{2}\cos\pi T - \frac{1}{2} - \frac{\vartheta_0}{\vartheta_1}}. \tag{3.4}$$

Therefore, design weight $1-\xi_{\rm t}$ is applied at the point A and $0.5\xi_{\rm t}$ at each of B and C under the symmetry assumption which we have imposed, see Figure 2. Due to the imposed discussion above, the a–p–o information matrix can be written in the form

$$\mathbf{M} = (1-\xi_{\rm t})\mathbf{M}_A + \tfrac{1}{2}\xi_{\rm t}\mathbf{M}_B + \tfrac{1}{2}\xi_{\rm t}\mathbf{M}_C, \tag{3.5}$$

recall Figure 2 and the regression equation (2.2). Relation (3.5) "breaks" the total information $\mathbf{M}$ into three "parts" which is exactly what we mean by "breaking into pieces" the total information or collecting piece–wise information. We need to know the information at points A, B and C or, equivalently, to check if the collected information at the points A, B and C is appropriately set. Therefore a decomposition of $\mathbf{M}$ into $\mathbf{M}_A$, $\mathbf{M}_B$ and $\mathbf{M}_C$ is installed. The accuracy of $\mathbf{M}_A$, $\mathbf{M}_B$ and $\mathbf{M}_C$ influences the total information $\mathbf{M}$.

There are many ways to split the a–p–o information matrix into three parts. Here, the most appropriate was chosen, using as a criterion the times t_A, t_B and t_C below and relations (3.6), (3.7) and (3.8) respectively. It is easy to see that, for any T,

$$2\pi t_A + \vartheta_2 = 0, \text{ i.e. } t_A = -\vartheta_2/2\pi, \ t_D = t_A - \tfrac{1}{2}, \text{ and}$$
$$\begin{aligned} t_B &= t_D + \tfrac{1}{2}T = \tfrac{1}{2}\left(-\tfrac{\vartheta_2}{\pi} - 1 + T\right), \\ t_C &= t_D - \tfrac{1}{2}T + 1 = \tfrac{1}{2}\left(-\tfrac{\vartheta_2}{\pi} + 1 - T\right). \end{aligned}$$

If we set $T = 1/3$ in the above relations –which seems appropriate in practice– the vectors corresponding to $W_i(t)$, $i = A, B, C$, recall (2.3), are:

$$W(t_A) = (1,\ \cos\vartheta_2,\ -\sin\vartheta_2)^{\rm T}, \tag{3.6}$$
$$W(t_B) = \left(1,\ \cos(\vartheta_2 + \tfrac{2\pi}{3}),\ -\sin(\vartheta_2 + \tfrac{2\pi}{3})\right)^{\rm T}, \tag{3.7}$$
$$W(t_C) = \left(1,\ \cos(\vartheta_2 - \tfrac{2\pi}{3}),\ -\sin(\vartheta_2 - \tfrac{2\pi}{3})\right)^{\rm T}. \tag{3.8}$$

We can therefore obtain $\mathbf{M}_i$, $i = A, B, C$ as

$$\mathbf{M}_i = W(t_i)W^{\rm T}(t_i), \quad i = A, B, C. \tag{3.9}$$

In turn, we prove the following Theorem.

Theorem 3.1. *The a–p–o information matrix as in* (3.5) *with* $T = 1/3$ *can be written as*

$$\mathbf{M}(\xi) = (1-\xi_{\rm t})A_1A_1^{\rm T} + \xi_{\rm t}A_2A_2^{\rm T} + \xi_{\rm t}A_3A_3^{\rm T}, \tag{3.10}$$

with vectors A_i*,* $i = 1, 2, 3$ *defined as*

$$A_1 = (1,\ \cos\vartheta_2,\ -\sin\vartheta_2)^{\rm T},\ \ A_2 = \left(1,\ -\tfrac{1}{2}\cos\vartheta_2,\ \tfrac{1}{2}\sin\vartheta_2\right)^{\rm T}, \quad \textit{and}$$

$$A_3 = \tfrac{\sqrt{3}}{2}\,(0,\ \sin\vartheta_2,\ \cos\vartheta_2)^{\rm T}.$$

Proof. From (3.5) and (3.9) with $T = \frac{1}{3}$ and $\delta = \pi(1-T) = \frac{2\pi}{3}$, we obtain (3.10) where

$$A_1 = (1,\ \cos\vartheta_2,\ -\sin\vartheta_2)^{\rm T},\ \ A_2 = \left(1,\ -\tfrac{1}{2}\cos\vartheta_2,\ \tfrac{1}{2}\sin\vartheta_2\right)^{\rm T}, \quad \text{and}$$

$$A_3 = \tfrac{\sqrt{3}}{2}\,(0,\ \sin\vartheta_2,\ \cos\vartheta_2)^{\rm T},$$

with $A_1 \perp A_2$ and $A_2 \perp A_3$, see Appendix for details. □

Our target is to evaluate $c^{\rm T}\mathbf{M}^{-}c$. Therefore, the following Lemma provides the evaluation of the desired quantity.

Lemma 3.2. *Let* $a = \frac{1}{2}(\cos\delta + 1)$*. The a–p–o information matrix* $\mathbf{M}(\xi_{\rm t})$ *can be written as*

$$\mathbf{M}(\xi_{\rm t}) = \mathbf{M}_0(\xi_{\rm t}) + a_{12}(B_1B_2^{\rm T} + B_2B_1^{\rm T}), \tag{3.11}$$

where $B_i = ||A_i||^{-1}A_i$*,* $i = 1, 2, 3$ *with* $\mathbf{M}_0 = a_1B_1B_1^{\rm T} + a_2B_2B_2^{\rm T} + a_3B_3B_3^{\rm T}$ *and*

$$\begin{aligned} a_1 &= 2(1-\xi_{\rm t}) + 2\xi_{\rm t}a^2, & a_2 &= 2\xi_{\rm t}(1-a)^2, \\ a_3 &= \xi_{\rm t}\sin^2\delta, & a_{12} &= 2a(1-a)\xi_{\rm t}. \end{aligned} \tag{3.12}$$

Proof. Recall (3.10). Consider the orthonormal vectors $B_i = ||A_i||^{-1}A_i$, $i = 1, 2, 3$, i.e.

$$B_1 = \tfrac{1}{\sqrt{2}}\,(1,\ \cos\vartheta_2,\ -\sin\vartheta_2)^{\rm T}, \quad B_2 = \tfrac{1}{\sqrt{2}}\,(1,\ -\cos\vartheta_2,\ \sin\vartheta_2)^{\rm T}, \quad \text{and}$$

$$B_3 = (0,\ \sin\vartheta_2,\ \cos\vartheta_2)^{\rm T}.$$

We can write the following decomposition of the vector

$$\begin{pmatrix} 1 \\ \cos\delta\cos\vartheta_2 \\ -\cos\delta\sin\vartheta_2 \end{pmatrix} = a\begin{pmatrix} 1 \\ \cos\vartheta_2 \\ -\sin\vartheta_2 \end{pmatrix} + (1-a)\begin{pmatrix} 1 \\ -\cos\vartheta_2 \\ \sin\vartheta_2 \end{pmatrix} = a\sqrt{2}B_1 + (1-a)\sqrt{2}B_2,$$

where $a = \frac{1}{2}(\cos\delta + 1)$. Applying the above to (3.10) we get

$$\mathbf{M}(\vartheta, \xi_{\rm t}) = \mathbf{M}_0 + a_{12}B_1B_2^{\rm T} + a_{12}B_2B_1^{\rm T} = \mathbf{M}_1 + a_{12}B_2B_1^{\rm T}, \tag{3.13}$$

with a_1, a_2, a_3 and a_{12} as in (3.12). See Appendix for details. □

We can express vectors B_i, $i = 1, 2, 3$ as linear combinations of the vectors c, d and e as follows,

$$B_1 = \lambda_1c + \lambda_2e,\ \ B_2 = \lambda_3c + \lambda_4e,\ \ B_3 = \tfrac{\sqrt{3}}{2}d,$$

where the defined vectors are

$$\begin{aligned} &c^{\rm T} = \tfrac{1}{\vartheta_0}\left(-\tfrac{\vartheta_1}{\vartheta_0},\ \cos\vartheta_2,\ -\sin\vartheta_2\right), \quad d^{\rm T} = (0,\ \sin\vartheta_2,\ \cos\vartheta_2) \quad \text{and} \\ &e^{\rm T} = \left(\tfrac{\vartheta_0}{\vartheta_1},\ \cos\vartheta_2,\ -\sin\vartheta_2\right) \end{aligned} \tag{3.14}$$

Due to the above Lemma 3.2 the a–p–o information matrix can be obtained as a summation of three components while the corresponding $c^{\rm T}\mathbf{M}(\xi)^{-1}c$ value can be evaluated through this decomposition. Indeed, we state and prove the following.

Theorem 3.3. *The a–p–o information matrix* $\mathbf{M}(\xi_t)$ *can be written as*

$$\mathbf{M}(\xi_t) = \lambda_c cc^{\mathrm{T}} + \lambda_d dd^{\mathrm{T}} + \lambda_e ee^{\mathrm{T}} \quad \textit{and} \tag{3.15}$$

$$\rho = c^{\mathrm{T}}\mathbf{M}^{-}(\xi_t)c = \frac{\lambda_c^{-1}}{1-\lambda_{ce}^2\lambda_c^{-1}\lambda_e^{-1}}. \tag{3.16}$$

Proof. Recall Lemma 3.2. It is clear that (3.15) holds considering (3.16) with $\lambda_c = (1-\xi)\lambda_1 + \xi\lambda_3$, $\lambda_d = \frac{3}{4}\xi_t$, $\lambda_e = (1-\xi)\lambda_2 + \xi\lambda_4$ and $\lambda_{ce} = (1-\xi)\lambda_1\lambda_2 + \xi\lambda_3\lambda_4$, where

$$\lambda_1 = \frac{1-g}{g^2+1}\vartheta_0, \quad \lambda_2 = \frac{(g+1)g}{g^2+1}, \quad \lambda_3 = -\frac{2g+1}{2(g^2+1)}\vartheta_0, \quad \lambda_4 = \frac{2g-g^2}{2(g^2+1)}, \quad g = \frac{\vartheta_1}{\vartheta_0}.$$

Our target now is to evaluate $\mathbf{M}^{-1} = \mathbf{M}^{-1}(\vartheta, \xi_t)$. Firstly,

$$\mathbf{M}_0^{-1} = a_1^{-1}B_1B_1^{\mathrm{T}} + a_2^{-1}B_2B_2^{\mathrm{T}} + a_3^{-1}B_3B_3^{\mathrm{T}}, \tag{3.17}$$

and secondly, see [8] among others,

$$\begin{aligned}\mathbf{M}_1^{-1} &= \left(\mathbf{M}_0 + a_{12}B_1B_2^{\mathrm{T}}\right)^{-1} = \mathbf{M}_0^{-1} - a_{12}\frac{\mathbf{M}_0^{-1}B_1B_2^{\mathrm{T}}\mathbf{M}_0^{-1}}{1+a_{12}B_2^{\mathrm{T}}\mathbf{M}_0^{-1}B_1} \\ &= \mathbf{M}_0^{-1} - a_{12}\mathbf{M}_0^{-1}B_1B_2^{\mathrm{T}}\mathbf{M}_0^{-1},\end{aligned}$$

because $B_2^{\mathrm{T}}\mathbf{M}_0^{-1}B_1 = 0$ due to (3.17) and the orthogonal vectors B_1, B_2, B_3. Thus,

$$\mathbf{M}^{-1} = \left(\mathbf{M}_1 + a_{12}B_2B_1^{\mathrm{T}}\right)^{-1} = \mathbf{M}_1^{-1} - a_{12}\frac{\mathbf{M}_1^{-1}B_2B_1^{\mathrm{T}}\mathbf{M}_1^{-1}}{1+a_{12}B_1^{\mathrm{T}}\mathbf{M}_1^{-1}B_2}. \tag{3.18}$$

It holds that $B_1^{\mathrm{T}}\mathbf{M}_0^{-1}B_2 = 0$ and $B_i^{\mathrm{T}}\mathbf{M}_0^{-1}B_i = a_i^{-1}$, $i = 1, 2$ due to (3.17), and thus

$$\begin{aligned}B_1^{\mathrm{T}}\mathbf{M}_1^{-1}B_2 &= B_1^{\mathrm{T}}(\mathbf{M}_0^{-1} - a_{12}\mathbf{M}_0^{-1}B_1B_2^{\mathrm{T}}\mathbf{M}_0^{-1})B_2 \\ &= B_1^{\mathrm{T}}\mathbf{M}_0^{-1}B_2 - a_{12}B_1^{\mathrm{T}}\mathbf{M}_0^{-1}B_1B_2^{\mathrm{T}}\mathbf{M}_0^{-1}B_2 = -a_{12}a_1^{-1}a_2^{-1},\end{aligned} \tag{3.19}$$

while

$$\begin{aligned}\mathbf{M}_1^{-1}B_2B_1^{\mathrm{T}}\mathbf{M}_1^{-1} &= \mathbf{M}_0^{-1}B_2B_1^{\mathrm{T}}\mathbf{M}_0^{-1} - \tfrac{a_{12}}{a_2}\mathbf{M}_0^{-1}B_1B_1^{\mathrm{T}}\mathbf{M}_0^{-1} - \\ &\quad \tfrac{a_{12}}{a_1}\mathbf{M}_0^{-1}B_2B_2^{\mathrm{T}}\mathbf{M}_0^{-1} + \tfrac{a_{12}^2}{a_1a_2}\mathbf{M}_0^{-1}B_1B_2^{\mathrm{T}}\mathbf{M}_0^{-1},\end{aligned} \tag{3.20}$$

see Appendix for details. Substituting (3.19) and (3.20) into (3.18) we get

$$\mathbf{M}^{-1} = \mathbf{M}_0^{-1} - a_{12}\mathbf{M}_0^{-1}B_1B_2^{\mathrm{T}}\mathbf{M}_0^{-1} - \frac{a_{12}}{1-\frac{a_{12}^2}{a_1a_2}}D, \quad \text{with}$$

$$D = \mathbf{M}_0^{-1}B_2B_1^{\mathrm{T}}\mathbf{M}_0^{-1} - \tfrac{a_{12}}{a_2}\mathbf{M}_0^{-1}B_1B_1^{\mathrm{T}}\mathbf{M}_0^{-1} - \tfrac{a_{12}}{a_1}\mathbf{M}_0^{-1}B_2B_2^{\mathrm{T}}\mathbf{M}_0^{-1} + \tfrac{a_{12}^2}{a_1a_2}\mathbf{M}_0^{-1}B_1B_2^{\mathrm{T}}\mathbf{M}_0^{-1}.$$

The analysis of (3.15) can be evaluated through an analysis similar to that of Lemma 3.2 or can be considered as special case of a_1, a_2, a_3 and a_{12} in (3.12). For (3.16), $\mathbf{M}^{-1}$ is evaluated as

$$\begin{aligned}\mathbf{M}^{-1} &= \mathbf{M}_0^{-1} - \lambda_{ce}\mathbf{M}_0^{-1}ce^{\mathrm{T}}\mathbf{M}_0^{-1} - \\ &\quad \lambda_{ce}\frac{\mathbf{M}_0^{-1} - \lambda_{ce}(\mathbf{M}_0^{-1}c)(e^{\mathrm{T}}\mathbf{M}_0^{-1})ec^{\mathrm{T}}(\mathbf{M}_0^{-1} - \lambda_{ce}\mathbf{M}_0^{-1}ce^{\mathrm{T}}\mathbf{M}_0^{-1})}{1-\lambda_{ce}^2(c^{\mathrm{T}}\mathbf{M}_0^{-1}c)(e^{\mathrm{T}}\mathbf{M}_0^{-1}e)}, \quad \text{and}\end{aligned}$$

$$\begin{aligned}c^{\mathrm{T}}\mathbf{M}^{-1}c &= c^{\mathrm{T}}\mathbf{M}_0^{-1}c + \lambda_{ce}^2\frac{(c^{\mathrm{T}}\mathbf{M}_0^{-1}c)^2(e^{\mathrm{T}}\mathbf{M}_0^{-1}e)}{1-\lambda_{ce}^2(c^{\mathrm{T}}\mathbf{M}_0^{-1}c)(e^{\mathrm{T}}\mathbf{M}_0^{-1}e)} \\ &= \frac{c^{\mathrm{T}}\mathbf{M}_0^{-1}c}{1-\lambda_{ce}^2(c^{\mathrm{T}}\mathbf{M}_0^{-1}c)(e^{\mathrm{T}}\mathbf{M}_0^{-1}e)},\end{aligned}$$

i.e. (3.16) holds with $\mathbf{M}_0 = \lambda_c cc^{\mathrm{T}} + \lambda_d dd^{\mathrm{T}} + \lambda_e ee^{\mathrm{T}}$ as $c^{\mathrm{T}}\mathbf{M}_0^{-1}c = \lambda_c^{-1}$ and $e^{\mathrm{T}}\mathbf{M}_0^{-1}e = \lambda_e^{-1}$. □

After some algebra we find that ρ is given by

$$\rho = (\tfrac{1}{3}\vartheta_0)^2 \frac{(4g+1)^2(1-\xi_t)+(2-g)^2\xi_t}{\xi_t(1-\xi_t)}, \quad g = \vartheta_1/\vartheta_0. \tag{3.21}$$

Therefore the approximate variance for the three point optimally–weighted truncated design is

$$\mathrm{V}_3 = \mathrm{Var}(c^{\mathrm{T}}\hat{\vartheta}) = \rho(\sigma^2/n). \tag{3.22}$$

Compare the reduction of (2.17) to (3.22). The corresponding design weight ξ_t can be evaluated from (3.4) with $T = 1/3$ as

$$\xi_t^* = \frac{\frac{1}{2}(g+1)}{\frac{1}{4}+g}. \tag{3.23}$$

Compare the reduction of (2.12) to (3.23). Thus the design measure still depends on the fraction ϑ_1/ϑ_0 which we are trying to estimate. If we wanted to construct an equally–weighted 3 point design in this truncated case, the design measure would be defined by $\rho = 2/3$ with corresponding approximate variance

$$\mathrm{V}_3^* = (\tfrac{2}{3}\vartheta_0^2)\left[2(g+1)^2+(2-g)^2\right](\sigma^2/n).$$

Now, consider a vector h which can be written as

$$h = vB_1 + (1-v)B_2, \quad v \in \mathbb{R}. \tag{3.24}$$

Then, from (2.14), we have $\mathbf{M}(\vartheta,\xi_t) = 2\xi_t B_1 B_1^{\mathrm{T}} + 2(1-\xi_t)B_2B_2^{\mathrm{T}}$, i.e.

$$\mathbf{M}^{-}(\vartheta,\xi_t) = \tfrac{1}{2}\xi_t^{-1}B_1B_1^{\mathrm{T}} + \tfrac{1}{2}(1-\xi_t)^{-1}B_2B_2^{\mathrm{T}}. \tag{3.25}$$

For any h as in (3.24), considering $\mathbf{M}^- = \mathbf{M}^-(\xi_t)$ as in (3.25), we find that

$$2h^{\mathrm{T}}\mathbf{M}^-(\xi_t)h = \frac{v^2}{\xi_t} + \frac{(1-v)^2}{1-\xi_t}. \tag{3.26}$$

Relation (3.26) gives the value of the criterion function for any ξ_t and it can easily be shown to be minimized when

$$\left|\frac{v}{1-v}\right| = \left|\frac{\xi_t}{1-\xi_t}\right|, \quad \text{i.e. } \frac{v}{1-v} = \frac{\xi_t}{1-\xi_t}. \tag{3.27}$$

Thus, for the particular h in which we are interested, namely

$$-\tfrac{\vartheta_1}{\vartheta_0}h = \left(1,\ -\tfrac{\vartheta_0}{\vartheta_1}\cos\vartheta_2,\ \tfrac{\vartheta_0}{\vartheta_1}\sin\vartheta_2\right),$$

we can obtain the design measure

$$v = \tfrac{1}{2}(1-\tfrac{\vartheta_0}{\vartheta_1}). \tag{3.28}$$

From (3.27) we have that either $\xi_t = v$ or $\xi_t = -v/(1-2v)$. From (3.28) we obtain

$$\xi_t = \tfrac{1}{2}(1-\vartheta_1/\vartheta_0) \quad \text{if } \vartheta_1/\vartheta_0 < 1. \tag{3.29}$$

Consider now the general problem of truncated daily time interval T and the observations as follows (recall Figure 2) allocating observations at the end points and the middle of $1-T$:

$$1-\xi_t \quad \text{at} \quad t_A = -\vartheta_2, \tag{3.30}$$

$$\tfrac{1}{2}\xi_t \quad \text{at} \quad t_B = -\vartheta_2 + 2\pi\cdot\tfrac{1}{2}(1-T) = -\vartheta_2+\delta, \tag{3.31}$$

$$\tfrac{1}{2}\xi_t \quad \text{at} \quad t_C = -\vartheta_2 - 2\pi\cdot\tfrac{1}{2}(1-T) = -\vartheta_2-\delta, \tag{3.32}$$

with $\delta = \pi(1-T)$. The corresponding vectors $W(t)$, recall (2.3), is then as in (3.10). Therefore the matrix $\mathbf{M} = \mathbf{M}(\vartheta,\xi_t)$ can be evaluated explicitly as

$$\mathbf{M} = (1-\xi_t)W(t_A)W(t_A)^{\mathrm{T}} + \tfrac{1}{2}\xi_t\left[W(t_B)W(t_B)^{\mathrm{T}} + W(t_C)W(t_C)^{\mathrm{T}}\right]. \tag{3.33}$$

The inverse of $\mathbf{M}(\xi_t)$ is needed as the general problem can be formulated as

$$\min_{\xi_t} \left\{ h^T \mathbf{M}(\xi_t)^{-1} h \right\}, \quad h = -\tfrac{\vartheta_1}{\vartheta_0^2} \left(1,\ -\tfrac{\vartheta_0}{\vartheta_1}\cos\vartheta_2,\ \tfrac{\vartheta_0}{\vartheta_1}\sin\vartheta_2\right)^T. \tag{3.34}$$

To obtain a c–optimal design ($c = h$), the following Theorem is proved.

Theorem 3.4. *The quantity $h^T\mathbf{M}^{-1}h$ is evaluated to be*

$$h^T\mathbf{M}^{-1}h = \frac{\vartheta_1^2}{2\vartheta_0^4}(1-a)^{-2}\left[\frac{(1-v)^2}{\xi_t} + \frac{(v-a)^2}{1-\xi_t}\right], \tag{3.35}$$

where $a = \frac{1}{2}(\cos\delta + 1)$, while the minimum of $h^T\mathbf{M}^{-1}h$ is obtained for

$$\xi_t = \frac{\frac{1}{2}\left(1 + \frac{\vartheta_0}{\vartheta_1}\right)}{\frac{1}{2} - \frac{1}{2}\cos\pi T + \frac{\vartheta_0}{\vartheta_1}}. \tag{3.36}$$

Proof. We want to minimize the quantity $h^T\mathbf{M}^{-1}h$, with

$$h = -\tfrac{\vartheta_1}{\vartheta_0^2}(1,\ -\tfrac{\vartheta_0}{\vartheta_1}\cos\vartheta_2,\ \tfrac{\vartheta_0}{\vartheta_1}\sin\vartheta_2).$$

The vector h can be written as

$$h = -\tfrac{\vartheta_1^2}{\vartheta_0^2}[vB_1 + (1-v)B_2], \quad v = \tfrac{1}{2}(1 - \vartheta_0/\vartheta_1).$$

Thus,

$$\begin{aligned} h^T\mathbf{M}^{-1}h &= \tfrac{\vartheta_1^2}{\vartheta_0^4}[vB_1 + (1-v)B_2]^T\mathbf{M}^{-1}[vB_1 + (1-v)B_2] \\ &= \tfrac{\vartheta_1^2}{\vartheta_0^4}\Big[v^2B_1^T\mathbf{M}^{-1}B_1 + v(1-v)\left(B_1^T\mathbf{M}^{-1}B_2 + B_2^T\mathbf{M}^{-1}B_1\right) + \\ &\quad (1-v)^2B_2^T\mathbf{M}^{-1}B_2\Big]. \end{aligned} \tag{3.37}$$

Recall (3.19) and let $\alpha = a_{12}/\left[1 - a_{12}^2/(a_1a_2)\right]$. We have

$$B_1^T\mathbf{M}^{-1}B_1 = a_1^{-1} - \alpha\left(-\frac{a_{12}}{a_2}\cdot\frac{1}{a_1a_1}\right) = \frac{1}{a_1} + \frac{a_{12}^2a_1^{-2}a_2^{-1}}{1 - a_{12}^2a_1^{-1}a_2^{-1}}, \tag{3.38}$$

while, similarly to (3.38), we get

$$B_2^T\mathbf{M}_1^{-1}B_2 = \frac{1}{a_2} + \frac{a_{12}^2a_1^{-1}a_2^{-2}}{1 - a_{12}^2a_1^{-1}a_2^{-1}}. \tag{3.39}$$

Moreover,

$$B_1^T\mathbf{M}^{-1}B_2 = -\frac{a_{12}}{a_1a_2} - \alpha\left(-\frac{a_{12}}{a_1^2a_2^2}\right) = -\frac{a_{12}}{a_1a_2} - \frac{a_{12}^3a_1^{-2}a_2^{-2}}{1 - a_{12}^2a_1^{-1}a_2^{-1}}, \tag{3.40}$$

while

$$B_2^T\mathbf{M}^{-1}B_1 = -\frac{a_{12}a_1^{-1}a_2^{-1}}{1 - a_{12}^2a_1^{-1}a_2^{-1}}. \tag{3.41}$$

See Appendix for details on (3.38), (3.40) and (3.41). Substituting (3.38), (3.39), (3.40) and (3.41) into (3.37), we have

$$\begin{aligned} h^T\mathbf{M}^{-1}h &= \tfrac{\vartheta_1^2}{\vartheta_0^4}\cdot\frac{v^2a_1^{-1} + (1-v)^2a_2^{-1} - 2v(1-v)a_{12}a_1^{-1}a_2^{-1}}{1 - a_{12}^2a_1^{-1}a_2^{-1}} \\ &= \tfrac{\vartheta_1^2}{\vartheta_0^4}\cdot\frac{v^2a_2 + (1-v)^2a_1 - 2v(1-v)a_{12}}{a_1a_2 - a_{12}^2} \\ &= \tfrac{\vartheta_1^2}{\vartheta_0^4}\cdot\frac{2(1-v)^2(1-\xi_t) + 2\xi_t[(1-a)v - a(1-v)]^2}{4\xi_t(1-\xi_t)(1-a)^2} \\ &= \tfrac{\vartheta_1^2}{2\vartheta_0^4}\cdot(1-a)^{-2}\left[\frac{(1-v)^2}{\xi_t} + \frac{(v-a)^2}{1-\xi_t}\right], \end{aligned}$$

and thus (3.35) has been proved.

Therefore, the minimum of (3.35), with respect to ξ_t, is obtained for ξ_t as in (3.33). The particular case of (3.33) discussed earlier, corresponds to (3.35) for $T = 1/3$. □

4 Conclusions

The target of this paper was to provide compact expressions for the a–p–o information matrix for the truncated cosinor model. This trigonometric model has been also discussed by [9]. The proved Theorems 3.1, 3.3 and 3.4 provide such expressions useful to trace the "flow" of information by "parts". Theorem 3.1 provide evidence that we can "break" the a–p–o information matrix into three pieces. Theorem 3.3 and 3.4 can be referred to practical cases (time, internal rhythm etc) where the collection of information can occur on "social hours". The information is related to the (inverse of) variance and the entropy (equals to $\log\det$ of the variance matrix). Thus, in real life problems and especially in Engineering and Bioinformatics, the provided uncertainty is easy to be adjusted in small "parts" of information rather than in total. Needles to say that although the proofs might be computationally tedious, the compact form of the result it is not. Moreover, when the data are collected, see [2], it is easy to see whether the "flow" of information is indeed collected correctly at the three stations A, B and C. For another real case biological data fitting the model, see the early work of [4] and [5] who are not using the design approach but a trigonometric regression one. Therefore, we derived a new method to "break" the total information (and uncertainty) into "parts" for the truncated cosinor model.

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Authors' contributions

All authors contributed equally and significantly in this research work. All authors read and approved the final manuscript.

Competing Interests

The authors declare that they have no competing interests.

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APPENDIX

Proof of (3.10)**.** We have

$$\begin{aligned}\mathbf{M}(\vartheta,\xi_t) &= (1-\xi_t)\begin{pmatrix}1 & \cos\vartheta_2 & -\sin\vartheta_2\\ \cos\vartheta_2 & \cos^2\vartheta_2 & -\cos\vartheta_2\sin\vartheta_2\\ -\sin\vartheta_2 & -\cos\vartheta_2\sin\vartheta_2 & \sin^2\vartheta_2\end{pmatrix}+\\ &\tfrac{1}{2}\xi_t\begin{pmatrix}1 & \cos(\vartheta_2+\delta) & -\sin(\vartheta_2+\delta)\\ \cos(\vartheta_2+\delta) & \cos^2(\vartheta_2+\delta) & -\sin(\vartheta_2+\delta)\cos(\vartheta_2+\delta)\\ -\sin(\vartheta_2+\delta) & -\sin(\vartheta_2+\delta)\cos(\vartheta_2+\delta) & \sin^2(\vartheta_2+\delta)\end{pmatrix}+\\ &\tfrac{1}{2}\xi_t\begin{pmatrix}1 & \cos(\vartheta_2-\delta) & -\sin(\vartheta_2-\delta)\\ \cos(\vartheta_2-\delta) & \cos^2(\vartheta_2-\delta) & -\sin(\vartheta_2-\delta)\cos(\vartheta_2-\delta)\\ -\sin(\vartheta_2-\delta) & -\sin(\vartheta_2-\delta)\cos(\vartheta_2-\delta) & \sin^2(\vartheta_2-\delta)\end{pmatrix},\end{aligned}$$

or

$$\begin{aligned}\mathbf{M}(\vartheta,\xi_t) &= (1-\xi_t)\begin{pmatrix}1\\ \cos\vartheta_2\\ -\sin\vartheta_2\end{pmatrix}(1,\ \cos\vartheta_2,\ -\sin\vartheta_2)+\\ &\xi_t\begin{pmatrix}1\\ \cos\delta\cos\vartheta_2\\ -\cos\delta\sin\vartheta_2\end{pmatrix}(1,\ \cos\delta\cos\vartheta_2,\ -\cos\delta\sin\vartheta_2)+\\ &\xi_t\begin{pmatrix}0\\ \sin\delta\sin\vartheta_2\\ \sin\delta\cos\vartheta_2\end{pmatrix}(0,\ \sin\delta\sin\vartheta_2,\ \sin\delta\cos\vartheta_2)\,.\end{aligned}$$

□

Proof of (3.13)**.** We have

$$\begin{aligned}\mathbf{M}(\vartheta,\xi_t) &= 2(1-\xi_t)B_1B_1^{\mathrm{T}}+\xi_t(\sin^2\delta)B_3B_3^{\mathrm{T}}+\\ &\quad \xi_t\left[\sqrt{2}aB_1+\sqrt{2}(1-a)B_2\right]\left[\sqrt{2}aB_1+\sqrt{2}(1-a)B_2\right]\\ &= 2(1-\xi_t)B_1B_1^{\mathrm{T}}+\xi_t(\sin^2\delta)B_3B_3^{\mathrm{T}}+\\ &\quad 2\xi_t\left[a^2B_1B_1^{\mathrm{T}}+a(1-a)B_1B_2^{\mathrm{T}}+(1-a)B_2B_1^{\mathrm{T}}+(1-a)^2B_2B_2^{\mathrm{T}}\right]\\ &= \underbrace{a_1B_1B_1^{\mathrm{T}}+a_2B_2B_2^{\mathrm{T}}+a_3B_3B_3^{\mathrm{T}}}_{\mathbf{M}_0}+a_{12}(B_1B_2^{\mathrm{T}}+B_2B_1^{\mathrm{T}})\\ &= \underbrace{\mathbf{M}_0+a_{12}B_1B_2^{\mathrm{T}}}_{\mathbf{M}_1}+a_{12}B_2B_1^{\mathrm{T}}\\ &= \mathbf{M}_1+a_{12}B_2B_1^{\mathrm{T}}.\end{aligned}$$

□

Proof of (3.20)**.** We have

$$\begin{aligned}\mathbf{M}_1^{-1}B_2B_1^{\mathrm{T}}\mathbf{M}_1^{-1} &= (\mathbf{M}_0^{-1} - a_{12}\mathbf{M}_0^{-1}B_1B_2^{\mathrm{T}}\mathbf{M}_0^{-1})B_2B_1^{\mathrm{T}} \times \\ &\quad (\mathbf{M}_0^{-1} - a_{12}\mathbf{M}_0^{-1}B_1B_2^{\mathrm{T}}\mathbf{M}_0^{-1}) \\ &= \mathbf{M}_0^{-1}B_2B_1^{\mathrm{T}}\mathbf{M}_0^{-1} - a_{12}\mathbf{M}_0^{-1}B_1B_2^{\mathrm{T}}\mathbf{M}_0^{-1}B_2B_1^{\mathrm{T}}\mathbf{M}_0^{-1} - \\ &\quad \mathbf{M}_0^{-1}B_2B_1^{\mathrm{T}}a_{12}\mathbf{M}_0^{-1}B_1B_2^{\mathrm{T}}\mathbf{M}_0^{-1} + \\ &\quad a_{12}^2\mathbf{M}_0^{-1}B_1B_2^{\mathrm{T}}\mathbf{M}_0^{-1}B_2B_1^{\mathrm{T}}\mathbf{M}_0^{-1}B_1B_2^{\mathrm{T}}\mathbf{M}_0^{-1} \\ &= \mathbf{M}_0^{-1}B_2B_1^{\mathrm{T}}\mathbf{M}_0^{-1} - \tfrac{a_{12}}{a_2}\mathbf{M}_0^{-1}B_1B_1^{\mathrm{T}}\mathbf{M}_0^{-1} - \\ &\quad \tfrac{a_{12}}{a_1}\mathbf{M}_0^{-1}B_2B_2^{\mathrm{T}}\mathbf{M}_0^{-1} + \tfrac{a_{12}^2}{a_1a_2}\mathbf{M}_0^{-1}B_1B_2^{\mathrm{T}}\mathbf{M}_0^{-1}.\end{aligned}$$

□

Proofs of (3.38)**,** (3.40) **and** (3.41)**.** We have respectively

$$\begin{aligned}B_1^{\mathrm{T}}\mathbf{M}^{-1}B_1 &= \underbrace{B_1^{\mathrm{T}}\mathbf{M}_0^{-1}B_1}_{a_1^{-1}} - a_{12}B_1^{\mathrm{T}}\mathbf{M}_0^{-1}B_1\underbrace{B_2^{\mathrm{T}}\mathbf{M}_0^{-1}B_1}_{0} - \alpha B_1^{\mathrm{T}}DB_1 \\ &= a_1^{-1} - \alpha\underbrace{B_1^{\mathrm{T}}\mathbf{M}_0^{-1}B_2}_{0}B_1^{\mathrm{T}}\mathbf{M}_0^{-1}B_1 + \alpha\tfrac{a_{12}}{a_2}\underbrace{B_1^{\mathrm{T}}\mathbf{M}_0^{-1}B_1}_{a_1^{-1}}\underbrace{B_1^{\mathrm{T}}\mathbf{M}_0^{-1}B_1}_{a_1^{-1}} + \\ &\quad \alpha\tfrac{a_{12}}{a_1}\underbrace{B_1^{\mathrm{T}}\mathbf{M}_0^{-1}B_2}_{0}B_2^{\mathrm{T}}\mathbf{M}_0^{-1}B_1 - \alpha\tfrac{a_{12}^2}{a_2a_1}B_1^{\mathrm{T}}\mathbf{M}_0^{-1}B_1\underbrace{B_2^{\mathrm{T}}\mathbf{M}_0^{-1}B_1}_{0}. \\ &= a_1^{-1} - \alpha\left(-\frac{a_{12}}{a_2}\cdot\frac{1}{a_1a_1}\right) = \frac{1}{a_1} + \frac{a_{12}^2a_1^{-2}a_2^{-1}}{1 - a_{12}^2a_1^{-1}a_2^{-1}},\end{aligned}$$

$$\begin{aligned}B_1^{\mathrm{T}}\mathbf{M}^{-1}B_2 &= \underbrace{B_1^{\mathrm{T}}\mathbf{M}_0^{-1}B_2}_{0} - a_{12}\underbrace{B_1^{\mathrm{T}}\mathbf{M}_0^{-1}B_1}_{a_1^{-1}}\underbrace{B_2^{\mathrm{T}}\mathbf{M}_0^{-1}B_2}_{a_2^{-1}} - \alpha B_1^{\mathrm{T}}DB_2 \\ &= -\frac{a_{12}}{a_1a_2} - \alpha\underbrace{B_1^{\mathrm{T}}\mathbf{M}_0^{-1}B_2}_{0}B_1^{\mathrm{T}}\mathbf{M}_0^{-1}B_2 + \alpha\tfrac{a_{12}}{a_2}B_1^{\mathrm{T}}\mathbf{M}_0^{-1}B_1\underbrace{B_1^{\mathrm{T}}\mathbf{M}_0^{-1}B_2}_{0} + \\ &\quad \alpha\tfrac{a_{12}}{a_1}\underbrace{B_1^{\mathrm{T}}\mathbf{M}_0^{-1}B_2}_{0}B_2^{\mathrm{T}}\mathbf{M}_0^{-1}B_2 - \alpha\tfrac{a_{12}^2}{a_2a_1}\underbrace{B_1^{\mathrm{T}}\mathbf{M}_0^{-1}B_1}_{a_1^{-1}}\underbrace{B_2^{\mathrm{T}}\mathbf{M}_0^{-1}B_2}_{a_2^{-1}} \\ &= -\frac{a_{12}}{a_1a_2} - \alpha\left(-\frac{a_{12}}{a_1^2a_2^2}\right) = -\frac{a_{12}}{a_1a_2} - \frac{a_{12}^3a_1^{-2}a_2^{-2}}{1 - a_{12}^2a_1^{-1}a_2^{-1}},\end{aligned}$$

and

$$\begin{aligned}B_1^{\mathrm{T}}\mathbf{M}^{-1}B_1 &= \underbrace{B_1^{\mathrm{T}}\mathbf{M}_0^{-1}B_1}_{a_1^{-1}} - a_{12}B_1^{\mathrm{T}}\mathbf{M}_0^{-1}B_1\underbrace{B_2^{\mathrm{T}}\mathbf{M}_0^{-1}B_1}_{0} - \alpha B_1^{\mathrm{T}}DB_1 \\ &= a_1^{-1} - \alpha\underbrace{B_1^{\mathrm{T}}\mathbf{M}_0^{-1}B_2}_{0}B_1^{\mathrm{T}}\mathbf{M}_0^{-1}B_1 + \alpha\tfrac{a_{12}}{a_2}\underbrace{B_1^{\mathrm{T}}\mathbf{M}_0^{-1}B_1}_{a_1^{-1}}\underbrace{B_1^{\mathrm{T}}\mathbf{M}_0^{-1}B_1}_{a_1^{-1}} +\end{aligned}$$

$$\alpha \frac{a_{12}}{a_1} \underbrace{B_1^{\mathrm{T}} \mathbf{M}_0^{-1} B_2}_{0} B_2^{\mathrm{T}} \mathbf{M}_0^{-1} B_1 - \alpha \frac{a_{12}^2}{a_2 a_1} B_1^{\mathrm{T}} \mathbf{M}_0^{-1} B_1 \underbrace{B_2^{\mathrm{T}} \mathbf{M}_0^{-1} B_1}_{0}.$$

$$= a_1^{-1} - \alpha \left(-\frac{a_{12}}{a_2} \cdot \frac{1}{a_1 a_1} \right) = \frac{1}{a_1} + \frac{a_{12}^2 a_1^{-2} a_2^{-1}}{1 - a_{12}^2 a_1^{-1} a_2^{-1}}, \qquad \square$$

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