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Amenability of a Class of Banach Algebras

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Abstract

In this paper we define a new multiplication on Banach algebra A using commute idempotent endomorphisms of A. Then we consider types of amenability and contractibility of A with this new multiplication. We will show that this new Banach algebra has better amenable properties than Banach algebra A.

Keywords: Bounded endomorphisms; Derivation; Inner derivation; Amenability; Contractibility; Banach algebra; Banach module

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1 Introduction

The notion of amenability in Banach algebras was introduced by Johnson in [1]. This notion also appeared in the work of A. Ya. Helemskii [2], which was published in the same year. Since then, amenability has become a major issue in Banach algebras theory. A Banach algebra is called amenable if its first cohomological groups $H^1(A, X^*)$ vanish for all dual Banach A-bimodules X^*. We recall that if A is a Banach algebra and X is a Banach A-bimodule, then X^*, the dual of X, has a natural A-bimodule structure defined by

$$\langle x, a \cdot x^* \rangle = \langle x \cdot a, x^* \rangle, \qquad \langle x, x^* \cdot a \rangle = \langle a \cdot x, x^* \rangle, \qquad (a \in A, x \in X, x^* \in X^*).$$

Such a Banach A-bimodule X^* is called a dual A-bimodule.
Let A be a Banach algebra and X be a Banach A-bimodule. A derivation $D : A \longrightarrow X$ is a linear map, always taken to be continuous, satisfying

$$D(ab) = D(a) \cdot b + a \cdot D(b) \qquad (a, b \in A).$$

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Given $x \in X$, the map $\delta_x(a) = a \cdot x - x \cdot a$ is a derivation on A which is called an inner derivation. For more details see [3] and [4].

A derivation $D : A \longrightarrow X$ is called approximately inner, if there exists a net $\{x_\alpha\} \subset X$ such that

$$D(a) = \lim_{\alpha}(a \cdot x_\alpha - x_\alpha \cdot a) \qquad (a \in A).$$

The limit being in norm. Note $\{x_\alpha\}$ in the above is not necessarily bounded. In [5], Ghahramani and Loy introduced generalized notions of amenability with the hope that it will yield Banach algebra without bounded approximate identity which nonetheless had a form of amenability. So far, however, all known approximate amenable Banach algebras have bounded approximate identities. They gave examples to show that for most of these new notions, the correspon-ding class of Banach algebras is larger than that of the classical amenable Banach algebra introduced by Johnson in [1]. According to their definition a Banach algebra A is approximately amenable if for any A-bimodule X, any derivation $D : A \longrightarrow X^*$ is approximately inner. A Banach algebra A is approximately contractible if every derivation from A into every Banach A-bimodule X is approximately inner.

Let A be a Banach algebra and X be a Banach A-bimodule. Let σ,τ be bounded endomorphisms of A, i.e. bounded homomorphisms from A into A. A linear mapping $D : A \longrightarrow X$ is a (σ, τ)-derivation, if

$$D(ab) = D(a) \cdot \sigma(b) + \tau(a) \cdot D(b),$$

for all $a, b \in A$. A linear map $D : A \longrightarrow X$ is a (σ, τ)-inner derivation, if there exists $x \in X$ such that $D(a) = x \cdot \sigma(a) - \tau(a) \cdot x$, for all $a \in A$. These derivations on Banach algebras are studied by Mirzavaziri and Moslehian in [6]. If every bounded (σ, τ)-derivation from A into X is (σ, τ)-inner, then A is said to be (σ, τ)-contractible Banach algebra. In particular, σ-contractibility is (σ, σ)-contractibility and the ordinary contractibility is indeed (id, id)-contractibility, where id denotes the identity map. Banach algebra A is called (σ, τ)-amenable, if for each Banach A-bimodule X, every (σ, τ)-derivation $D : A \longrightarrow X^*$ is (σ, τ)-inner. Banach algebra A is called (σ, τ)-approximately contractible, if for each Banach A-bimodule X, and for each bounded (σ, τ)-derivation $D : A \longrightarrow X$, there exists a net $(x_\alpha) \subseteq X$ such that $D(a) = lim_\alpha x_\alpha \cdot \sigma(a) - \tau(a) \cdot x_\alpha$, for all $a \in A$.

Let A be a Banach algebra over $\mathbb{C}$ and $\varphi : A \to \mathbb{C}$ be a character on A, that is, an algebra homomorphism from A into $\mathbb{C}$ and let $\Phi(A)$ denote the character space of A (the set of all characters on A). In [7], Monfared introduced the notion of character amenable Banach algebra, which requires continuous derivations from A into dual Banach A-bimodules to be inner, but only those modules are concerned where either of the left or right module action is defined by characters on A, that is,

$$a \cdot x = \varphi(a)x, \qquad x \cdot a = \varphi(a)x, \qquad (a \in A,\ x \in X).$$

As such character amenability is weaker than the classical amenability introduced by Johnson in [1], all amenable Banach algebras are character amenable.

Now, let A be a Banach algebra and $A\widehat{\otimes}A$ be the projective tensor product of A and A. The product map on A extends to a map $\pi_A : A\widehat{\otimes}A \longrightarrow A$ determined by $\pi_A(a \otimes b) = ab$, for all $a, b \in A$. The projective tensor product $A\widehat{\otimes}A$ becomes a Banach A-bimodule with the following usual module actions:

$$a \cdot (b \otimes c) = ab \otimes c, \qquad (b \otimes c) \cdot a = b \otimes ca, \qquad (a, b, c \in A).$$

Obviously, by above actions, π_A becomes an A-bimodule homomorphism. The dual map π_A^* is also A-bimodule homomorphism. A Banach algebra A is called biprojective, if there exists a bounded A-bimodule homomorphism $\rho : A \longrightarrow A\widehat{\otimes}A$ such that $\pi \circ \rho = I_A$. Also A is said to be biflat if π_A^* has a left inverse as a bounded A-bimodule homomorphism.

An element $m \in A\widehat{\otimes}A$ is called a diagonal for A, if

$$a \cdot m = m \cdot a, \qquad a \cdot \pi_A(m) = a, \qquad (a \in A).$$

A virtual diagonal for A is an element $M \in (A\widehat{\otimes}A)^{**}$ such that for each $a \in A$ we have,

$$a \cdot M = M \cdot a, \qquad \pi_A^{**}(M) \cdot a = a.$$

It is known that every contractible Banach algebra is unital, biprojective and has a diagonal, ([8], Theorem 2.8.48). Also every amenable Banach algebra is biflat and has a bounded approximate identity, ([9], Proposition 2.2.1), and it has a virtual diagonal, ([9], Theorem 2.2.4).

To complete this section we recall that a Banach algebra A is said to be semisimple if $rad(A) = o$, where $rad(A)$ is the Jacobson radical of A. Also an involution on Banach algebra A is a map $* : A \to A$ such that for each $a, b \in A$ and $\lambda, \mu \in \mathbb{C}$,

(i) $a^{**} = a$
(ii) $(\lambda a + \mu b)^* = \overline{\lambda} a^* + \overline{\mu} b^*$
(iii) $(ab)^* = b^* a^*$

A Banach algebra A with an involution is called a $*$-Banach algebra.

This paper has been organized as follows. In the next section, using the commute idempotent endomorphisms σ and τ on a Banach algebra A, we define a new multiplication under which the Banach algebra structure of A is preserved. This new Banach algebra is denoted by ${}_\sigma A_\tau$ and existence of identity is discussed in this section. In Section 3, contractibility, amenability, etc., are discussed for this new Banach algebra ${}_\sigma A_\tau$ in relation with the corresponding properties in the original Banach algebra A. In Section 4, some other aspects, viz., ${}_\sigma A_\tau$ as a semisimple Banach algebra, as a $*$-algebra, contractibility of $\frac{A}{I}$, where I is a closed ideal of A, are also discussed with many new ideas.

2 Banach Algebra ${}_\sigma A_\tau$ and Existence of Identity Element

Let A be a Banach algebra and σ, τ be commute idempotent endomorphisms of A, i.e., $\sigma \circ \tau = \tau \circ \sigma$, such that $\| \sigma \| \leq 1$, $\| \tau \| \leq 1$. We define a new multiplication on A as follows,

$$a \cdot b = \sigma(a)\, \tau(b), \qquad (a, b \in A).$$

A number of examples of commute idempotent endomorphisms are listed below:

(i) If $\sigma = id_A$ and τ is an idempotent endomorphism of A, then σ, τ are commute idempotent endomorphisms of A.

(ii) If σ is an idempotent endomorphism of A and $\tau = \sigma^n, n \geq 1$ then σ, τ are commute idempotent endomorphisms of A.

(iii) Let b be an idempotent element in A. Then $\sigma = L_b$ and $\tau = R_b$ are commute idempotent endomorphisms of A, where $L_b(a) = ba$ and $R_b(a) = ab$, for each $a \in A$.

(iv) Let X be a compact Hausdorff space and suppose $\varphi, \psi : X \longrightarrow X$ are commute idempotent local homeomorphisms. Define $\sigma, \tau : C(X) \longrightarrow C(X)$ by $\sigma(f) = f \circ \varphi$ and $\tau(f) = f \circ \psi$ for all $f \in C(X)$. Then σ and τ are commute idempotent endomorphisms of $C(X)$.

It is easy to see that $(A, \cdot)$ becomes a Banach algebra. We denote this new Banach algebra by ${}_\sigma A_\tau$. We shall omit the letter σ (τ), when $\sigma = id_A$ $(\tau = id_A)$, where $id_A : A \to A$ is the identity operator.

Remark 2.1*.* Throughout this paper we shall assume that σ and τ are commute idempotent endomorphisms, (i.e., $\sigma \circ \sigma = \sigma$ and $\tau \circ \tau = \tau$), such that $\| \sigma \| \leq 1$ and $\| \tau \| \leq 1$. Additional assumptions will be said in its place.

In the following we will study the existence of identity in ${}_\sigma A_\tau$. Note that all propositions will express for σ. It is obvious that all these propositions will be true for τ by using a similar proof.

Lemma 2.1. *Let A be a unital Banach algebra (with unit e) and σ be an idempotent endomorphism of A with dense range. Then $\sigma(e) = e$.*

Proof. Let $(a_\alpha)_{\alpha \in I} \subseteq A$ be a net such that $\lim_\alpha \sigma(a_\alpha) = e$. Since σ is continuous idempotent homomorphism, we have

$$\lim_\alpha \sigma(a_\alpha) = \sigma(e)$$

and therefore $\sigma(e) = e$. □

Proposition 2.1. *Let A be a Banach algebra with right identity e and σ, τ be two idempotent endomorphisms of A with dense range. If σ is $1-1$ and ${}_\sigma A_\tau$ has a left identity $\acute{e}$, then $\acute{e} = e$.*

Proof. Since $\overline{\sigma(A)} = A$ and $\overline{\tau(A)} = A$ by previous lemma we have $\sigma(e) = e$ and $\tau(e) = e$. Now for each $a \in A$, by hypothesis $\acute{e} \cdot a = a$. So for $e \in A$, $\acute{e} \cdot e = e$, too. By multiplication on ${}_\sigma A_\tau$ we have,

$$\begin{aligned} \sigma(\acute{e})\, \tau(e) &= e \\ &\Rightarrow \sigma(\acute{e})\, e = e \\ &\Rightarrow \sigma(\acute{e}) = e \\ &\Rightarrow \sigma(\acute{e}) = \sigma(e) \\ &\Rightarrow \acute{e} - e \in \ker \sigma = \{0\}. \end{aligned}$$

So, $\acute{e} = e$ and the proof is complete. □

Corollary 2.2. *Let A be a unital Banach algebra with identity e and σ, τ be two idempotent endomorphisms of A with dense range. If σ and τ are $1-1$ and ${}_\sigma A_\tau$ has an identity $\acute{e}$, then $\acute{e} = e$.*

Proposition 2.2. *Let A be a Banach algebra with left identity e. If σ is an idempotent endomorphism of A with dense range, then e is a left identity for ${}_\sigma A$.*

Proof. By lemma 2, we have $\sigma(e) = e$. Now for each $a \in A$,

$$e \cdot a = \sigma(e)\, a = ea = a.$$

□

Proposition 2.3. *Let A be a Banach algebra with left identity e. Then e is a left identity for $\sigma({}_\sigma A)$.*

Proof. For each $a \in A$ we have,

$$e \cdot \sigma(a) = \sigma(e)\, \sigma(a) = \sigma(ea) = \sigma(a).$$

□

Proposition 2.4. *Let A be a Banach algebra and e be a right identity for $\sigma(A)$. If $\overline{\tau(A)} = A$, then e is a right identity for $\sigma({}_{\sigma}A_{\tau})$.*

Proof. By lemma 2 we have $\tau(e) = e$. So for each $a \in A$ we have

$$\sigma(a) \cdot e = \sigma(\sigma(a))\,\tau(e) = \sigma(a)\,e = \sigma(a).$$

□

Next, assume that A is a complex Banach space which has dimension at least 2 and let $0 \neq \varphi \in Ball(A^*)$. Define a multiplication on A by

$$a * b = \varphi(a)b \qquad (a, b \in A).$$

This multiplication evidently makes A into a Banach algebra denoted by A_φ, which is called the ideally factored algebra associated to φ, [10]. It is easy to see that A_φ has left identity e which is that element in A such that $\varphi(e) = 1$, while it has not right approximate identity. Suppose that $\sigma : A_\varphi \to A_\varphi$ be defined by $\sigma(a) = \varphi(a)\,e$. Then σ is the only idempotent endomorphism of A_φ.

In the following we show that with the only homomorphism σ of A_φ we can define only one new Banach algebra from A_φ. First we consider the product in Banach algebra $(A_\varphi)_\sigma$. For each $a, b \in (A_\varphi)_\sigma$ we have,

$$a \cdot b = a * \sigma(b) = \varphi(a)\,\sigma(b) = \varphi(a)\,\varphi(b)\,e.$$

Also the product in Banach algebra ${}_{\sigma}(A_\varphi)$ is as follows,

$$a \cdot b = \sigma(a) * b = \varphi(\sigma(a))\,b = \varphi(\varphi(a)\,e)\,b = \varphi(a)\,\varphi(e)\,e = \varphi(a)\,e = a * b,$$

$a, b \in {}_{\sigma}(A_\varphi)$. Therefore, the Banach algebra ${}_{\sigma}(A_\varphi)$ is exactly the Banach algebra A_φ.

The product in Banach algebra ${}_{\sigma}(A_\varphi)_\sigma$ is as follows,

$$a \cdot b = \sigma(a) * \sigma(b) = \varphi(\sigma(a))\,\sigma(b) = \varphi(a)\,\varphi(e)\,\varphi(b)\,e = \varphi(a)\,\varphi(b)\,e,$$

$a, b \in {}_{\sigma}(A_\varphi)_\sigma$, which shows that The Banach algebra ${}_{\sigma}(A_\varphi)_\sigma$ is exactly the Banach algebra $(A_\varphi)_\sigma$.

Now we show that the condition in proposition 6 is not necessary. First note that it is easy to see that the Banach algebra $(A_\varphi)_\sigma$ has not left and right identity. We prove that e is an identity for $\sigma\left((A_\varphi)_\sigma\right)$, where e is the left identity in A_φ. Let $a \in (A_\varphi)_\sigma$. So we have,

$$e \cdot \sigma(a) = \varphi(e)\,\varphi(\sigma(a))\,e = \varphi(\varphi(a)\,e)\,e = \varphi(a)\,e = \sigma(a).$$

Also,

$$\sigma(a) \cdot e = \varphi(\sigma(a))\,\varphi(e)\,e = \varphi(\varphi(a)\,e)\,e = \varphi(a)\,e = \sigma(a),$$

which shows that e is an identity for $\sigma\left((A_\varphi)_\sigma\right)$. Although A_φ has not right identity.

3 Contractibility and Amenability of ${}_{\sigma}A_{\tau}$

In this section we consider the relations between contractibility and amenability of Banach algebra A and ${}_{\sigma}A_{\tau}$. We start this section with the following lemma.

Lemma 3.1. *Let A be a Banach algebra. suppose σ is an idempotent endomorphism with dense range and τ is an idempotent epimorphism of A, (i.e., a surjective endomorphism of A). Then $\varphi : A \to {}_{\sigma}A_{\tau}$ defined by $\varphi(a) = \sigma(\tau(a))$ is a continuous idempotent homomorphism on A which has dense range.*

Proof. It is easy to see that φ is an idempotent homomorphism. Let $a \in {}_{\sigma}A_{\tau}$, since $\overline{\sigma(A)} = A$ there exists a net $(b_\alpha)_{\alpha \in I} \subseteq A$ such that $\lim_\alpha \sigma(b_\alpha) = a$. Also since τ is a surjective map , for each $\alpha \in I$, there exists $a_\alpha \in A$ such that $\tau(a_\alpha) = b_\alpha$. So we have,

$$a = \lim_{\alpha} \sigma(b_\alpha) = \lim_{\alpha} \sigma(\tau(a_\alpha)) = \lim_{\alpha} \varphi(a_\alpha) \qquad (a \in A),$$

which shows that $\overline{\varphi(A)} = {}_{\sigma}A_{\tau}$. □

Corollary 3.2. *Let A be a Banach algebra and σ, τ be two idempotent epimorphisms of A. Then $\varphi : A \to {}_{\sigma}A_{\tau}$ defined by $\varphi(a) = \sigma(\tau(a))$ is a surjective idempotent homomorphism on A.*

Proposition 3.1. *Let A be a Banach algebra, σ be an idempotent endomorphism with dense range and τ be an idempotent epimorphism of A. If any of the following conditions hold, then ${}_{\sigma}A_{\tau}$ is contractible.*

$i)$ A is τ-contractible.
$ii)$ A is σ-contractible.
$iii)$ A is (τ, σ)-contractible.
$iv)$ A is (σ, τ)-contractible.

Proof. Throughout this proof we assume that φ is the idempotent homomorphism which is defined in Lemma 8, $i.e.$ $\varphi : A \to {}_{\sigma}A_{\tau}$ defined by $\varphi(a) = \sigma(\tau(a))$.

$i)$ Let X be a Banach ${}_{\sigma}A_{\tau}$-bimodule and $D : {}_{\sigma}A_{\tau} \to X$ be a continuous derivation. Then $(X, *)$ is an A-bimodule with the following module actions:

$$a * x = \sigma(a) \cdot x \quad , \quad x * a = x \cdot \sigma(a) \qquad (a \in A, x \in X).$$

Since D is a derivation on ${}_{\sigma}A_{\tau}$, therefore $D \circ \varphi : A \to (X, *)$ is a τ derivation because,

$$\begin{aligned} D \circ \varphi(ab) &= D(\varphi(a) \cdot \varphi(b)) \\ &= D(\varphi(a)) \cdot \varphi(b) + \varphi(a) \cdot D(\varphi(b)) \\ &= D \circ \varphi(a) \cdot \sigma(\tau(b)) + \sigma(\tau(b)) \cdot D \circ \varphi(b) \\ &= D \circ \varphi(a) * \tau(b) + \tau(b) * D \circ \varphi(b) \qquad (a, b \in A). \end{aligned}$$

Since A is τ-contractible, there exists $x \in X$ such that

$$D \circ \varphi(a) = \tau(a) * x - x * \tau(a) \qquad (a \in A).$$

Thus

$$\begin{aligned} D(\varphi(a)) &= (D\circ\varphi)(a) \\ &= \tau(a) * x - x * \tau(a) \\ &= \sigma(\tau(a))\cdot x - x\cdot\sigma(\tau(a)) \\ &= \varphi(a)\cdot x - x\cdot\varphi(a) \qquad (a\in A). \end{aligned}$$

Now for each $b \in A$, by previous lemma, there exists a net $(a_\alpha) \subseteq A$ such that $b = \lim_\alpha \varphi(a_\alpha)$. So we have,

$$\begin{aligned} D(b) &= D\left(\lim_\alpha \varphi(a_\alpha)\right) \\ &= \lim_\alpha D(\varphi(a_\alpha)) \\ &= \lim_\alpha \varphi(a_\alpha)\cdot x - x\cdot\varphi(a_\alpha) \\ &= b\cdot x - x\cdot b \qquad (b\in A), \end{aligned}$$

which shows that ${}_\sigma A_\tau$ is contractible.

ii) For a Banach ${}_\sigma A_\tau$-bimodule X, it is easy to see that $(X,*)$ is an A-bimodule with the following module actions:

$$a * x = \tau(a)\cdot x \quad , \quad x * a = x\cdot\tau(a) \qquad (a\in A, x\in X).$$

The remaining argument is similar to (i).

iii) Let X be a Banach ${}_\sigma A_\tau$-bimodule and $D : {}_\sigma A_\tau \to X$ be a continuous derivation. Then $(X,*)$ is an A-bimodule with the following module actions:

$$a * x = \sigma(a)\cdot x \quad , \quad x * a = x\cdot\tau(a) \qquad (a\in A, x\in X).$$

Since D is a derivation on ${}_\sigma A_\tau$, so for each $a, b \in A$ we have,

$$\begin{aligned} D\circ\varphi(ab) &= D(\varphi(a)\cdot\varphi(b)) \\ &= D(\varphi(a))\cdot\varphi(b) + \varphi(a)\cdot D(\varphi(b)) \\ &= D(\varphi(a))\cdot\sigma(\tau(b)) + \sigma(\tau(a))\cdot D(\varphi(b)) \\ &= D(\varphi(a))\cdot\tau(\sigma(b)) + \sigma(\tau(a))\cdot D(\varphi(b)) \\ &= D\circ\varphi(a) * \sigma(b) + \tau(a) * D\circ\varphi(b). \end{aligned}$$

Thus $D\circ\varphi : A \to (X,*)$ is a (τ,σ)- derivation. Since A is (τ,σ)-contractible, there exists $x \in X$ such that

$$D\circ\varphi(a) = \tau(a) * x - x * \sigma(a) \qquad (a\in A).$$

So for each $a \in A$,

$$\begin{aligned} D(\varphi(a)) &= (D\circ\varphi)(a) \\ &= \tau(a) * x - x * \sigma(a) \\ &= \sigma(\tau(a))\cdot x - x\cdot\tau(\sigma(a)) \\ &= \varphi(a)\cdot x - x\cdot\varphi(a). \end{aligned}$$

Now by lemma 8, for each $b \in A$ there exists a net $(a_\alpha) \subseteq A$ such that $b = \lim_\alpha \varphi(a_\alpha)$. So we have,

$$\begin{aligned} D(b) &= D\left(\lim_\alpha \varphi(a_\alpha)\right) \\ &= \lim_\alpha D(\varphi(a_\alpha)) \\ &= \lim_\alpha \varphi(a_\alpha)\cdot x - x\cdot\varphi(a_\alpha) \\ &= b\cdot x - x\cdot b \qquad (b\in A), \end{aligned}$$

which shows that ${}_{\sigma}A_{\tau}$ is contractible.

$iv)$ It is similar to (iii). □

Example 3.3. *Let G be a locally compact group, $A = L^1(G)$, the group algebra of G, and σ be a bounded dense range endomorphism of $L^1(G)$. It is known that $L^1(G)$ is σ-contractible if and only if G is finite [11]. Therefore, by above Proposition, for each idempotent epimorphism τ of $L^1(G)$, the new Banach algebra ${}_{\sigma}(L^1(G))_{\tau}$ is contractible. In particular, ${}_{\sigma}(L^1(G))$ is a contractible Banach algebra.*

Example 3.4. *Let G be a locally compact group, $A = M(G)$, the measure algebra of G, and σ be a bounded dense range endomorphism of $M(G)$. It is known that $M(G)$ is σ-contractible if and only if G is finite [11]. Therefore, by above Proposition, for each idempotent epimorphism τ of $M(G)$, the new Banach algebra ${}_{\sigma}(M(G))_{\tau}$ is contractible. In particular, ${}_{\sigma}(M(G))$ is a contractible Banach algebra.*

Corollary 3.5. *Let A be a Banach algebra, σ be an idempotent endomorphism with dense range and τ be an idempotent epimorphism of A. If any of the conditions stated in the previous proposition occurs, then all the following hold:*

$i)$ ${}_{\sigma}A_{\tau}$ has an identity.

$ii)$ ${}_{\sigma}A_{\tau}$ has a diagonal.

$iii)$ ${}_{\sigma}A_{\tau}$ is biprojective.

Corollary 3.6. *Let A be a Banach algebra, σ be an idempotent endomorphism with dense range and τ be an idempotent epimorphism of A. If A be able to have one of these properties: τ-amenability, σ-amenability, (τ, σ)-amenability, (σ, τ)-amenability, then all the following hold:*

$i)$ ${}_{\sigma}A_{\tau}$ is amenable.

$ii)$ ${}_{\sigma}A_{\tau}$ has a bounded approximate identity.

$iii)$ ${}_{\sigma}A_{\tau}$ has a Virtual diagonal.

$iv)$ ${}_{\sigma}A_{\tau}$ is biflat.

Now let A_{φ} be the ideally factored algebra associated to φ, where $0 \neq \varphi \in Ball(A^*)$, as notation in previous section. Also let $\sigma : A_{\varphi} \to A_{\varphi}$ with definition $\sigma(a) = \varphi(a)\, e$, be the only homomorphism of A_{φ}. In [12] we showed that A_{φ} is σ-contractible Banach algebra. On the other hand in the previous section we see that $(A_{\varphi})_{\sigma}$ has not an identity. So clearly $(A_{\varphi})_{\sigma}$ is not contractible. Note that this does not contradict with the proposition 10, because it is clear that $\overline{\sigma(A)} \neq A$.

Proposition 3.2. *Let A be a Banach algebra and σ, τ be two idempotent epimorphisms of A. If any of the following conditions hold, then ${}_{\sigma}A_{\tau}$ is approximately contractible.*

$i)$ A is τ-approximately contractible.

$ii)$ A is σ-approximately contractible.

$iii)$ A is (τ, σ)-approximately contractible.

$iv)$ A is (σ, τ)-approximately contractible.

Proof. Throughout this proof we assume that φ is the idempotent homomorphism which is defined in lemma 8, $i.e.$ $\varphi : A \to {}_{\sigma}A_{\tau}$ defined by $\varphi(a) = \sigma(\tau(a))$.

$i)$ Let X be a Banach ${}_{\sigma}A_{\tau}$-bimodule and $D : {}_{\sigma}A_{\tau} \to X$ be a continuous derivation. Then $(X, *)$ is an A-bimodule with the following module actions:

$$a * x = \sigma(a) \cdot x \quad , \quad x * a = x \cdot \sigma(a) \qquad (a \in A, x \in X).$$

It is easy to see that $D\circ\varphi : A \to (X,*)$ is a continuous $\tau-$derivation. Since A is $\tau-$approximately contractible, there exists a net $(x_\alpha) \in X$ such that

$$D \circ \varphi (a) = \lim_\alpha \tau (a) * x_\alpha - x_\alpha * \tau (a) \qquad (a \in A) .$$

Now by Corollary 9, for each $b \in A$ there exists $a \in A$ such that $b = \varphi (a)$. So we have,

$$\begin{aligned} D (b) &= D (\varphi (a)) \\ &= \lim_\alpha \tau (a) * x_\alpha - x_\alpha * \tau (a) \\ &= \lim_\alpha \varphi (a) \cdot x_\alpha - x_\alpha \cdot \varphi (a) \\ &= \lim_\alpha b \cdot x_\alpha - x_\alpha \cdot b \qquad\qquad (b \in A) , \end{aligned}$$

which shows that ${}_\sigma A_\tau$ is approximately contractible.

The same argument as in proposition 10 shows that if any of conditions ii, iii and iv holds, then ${}_\sigma A_\tau$ is approximately contractible. □

Corollary 3.7. *Let A be a Banach algebra and σ,τ be two idempotent epimorphism of A. If any of the conditions stated in the previous proposition occur, then ${}_\sigma A_\tau$ has a left and right approximate identity.*

Proof. It is clear by ([12], Proposition 2.1). □

Corollary 3.8. *Let A be a Banach algebra and σ,τ be two idempotent epimorphism of A. If A be able to have one of these properties: τ-approximate amenability, σ-approximate amenability, (τ,σ)-approximate amenability, (σ,τ)-approximate amenability, then all the following hold.*

$i)$ ${}_\sigma A_\tau$ is approximately amenable.

$ii)$ ${}_\sigma A_\tau$ has a left and right approximate identity.

$iii)$ $\overline{{}_\sigma A_\tau^2} = {}_\sigma A_\tau$.

Proof. It is clear by proposition 13 and ([5], lemma 2.2). □

Proposition 3.3. *Let A be a Banach algebra, σ be an idempotent endomorphism with dense range and τ be an idempotent epimorphism of A. If A is character contractible, then ${}_\sigma A_\tau$ is character contractible.*

Proof. Throughout this proof we assume that φ is that idempotent homomorphism which is defined in lemma 8, $i.e.$ $\varphi : A \to {}_\sigma A_\tau$ with definition $\varphi (a) = \sigma (\tau (a))$ ($a \in A$).

Suppose that $\psi \in \Phi ({}_\sigma A_\tau)$, the character space of ${}_\sigma A_\tau$, and X is a Banach $({}_\sigma A_\tau, \psi)$-bimodule, that means the module actions are as follow,

$$a \cdot x = \psi (a) x, \qquad x \cdot a = x\psi (a) \qquad (a \in A, x \in X) .$$

Let $D : {}_\sigma A_\tau \to X$ be a continuous derivation. Then $(X,*)$ is an $(A, \psi \circ \varphi)$-bimodule with the following module actions:

$$a * x = \psi (\varphi (a)) x, \qquad x * a = x\psi (\varphi (a)) \quad (a \in A, x \in X) .$$

So $D \circ \varphi : A \to X$ is a continuous derivation because for each $a, b \in A$ we have,

$$\begin{aligned} D \circ \varphi (ab) &= D (\varphi (a) \varphi (b)) \\ &= D (\varphi (a)) \cdot \varphi (b) + \varphi (a) \cdot D (\varphi (b)) \\ &= D (\varphi (a)) \psi (\varphi (b)) + \psi (\varphi (a)) D (\varphi (b)) \\ &= D \circ \varphi (a) * b + a * D \circ \varphi (b) . \end{aligned}$$

Since A is character contractible, so there exists $x \in X$ such that,

$$D \circ \varphi (a) = a * x - x * a.$$

Thus for each $a \in A$ we have,

$$D (\varphi (a)) = Do\varphi (a) = a * x - x * a = \psi (\varphi (a)) x - x\psi (\varphi (a)) .$$

Now by lemma 8, since $\overline{\varphi (A)} = {}_\sigma A_\tau$, for each $b \in A$ there exists a net $(a_\alpha) \subseteq A$ such that $b = \lim_\alpha \varphi (a_\alpha)$. So we have,

$$\begin{aligned} D (b) &= D \left(\lim_\alpha \varphi (a_\alpha) \right) \\ &= \lim_\alpha D (\varphi (a_\alpha)) \\ &= \lim_\alpha \psi (\varphi (a_\alpha)) x - x\psi (\varphi (a_\alpha)) \\ &= \psi (b) x - x\psi (b) \\ &= b \cdot x - x \cdot b \qquad (b \in A) , \end{aligned}$$

which shows that ${}_\sigma A_\tau$ is character contractible. □

4 Some other Properties

Proposition 4.1. *Let A be a Banach algebra and σ, τ be two idempotent endomorphisms of A with dense range. Then $\Phi ({}_\sigma A_\tau) \subseteq \Phi (A)$.*

Proof. Let $\varphi \in \Phi ({}_\sigma A_\tau)$. So for each $a, b \in A$ we have,

$$\varphi (a \cdot b) = \varphi (a) \varphi (b) \qquad \Rightarrow \varphi (\sigma (a) \tau (b)) = \varphi (a) \varphi (b) .$$

Now let $a, b \in A$, since $\overline{\sigma (A)} = A$ and $\overline{\tau (A)} = A$, there exist nets (a_α) and (b_β) in A such that

$$\lim_\alpha \sigma (a_\alpha) = a \quad , \quad \lim_\beta \tau (b_\beta) = b.$$

On the other hand since σ and τ are idempotents, so we have

$$\lim_\alpha \sigma (a_\alpha) = \sigma (a) \quad , \quad \lim_\beta \tau (b_\beta) = \tau (b) .$$

Thus

$$\begin{aligned} \varphi (ab) &= \varphi \left(\lim_\alpha \sigma (a_\alpha) \;\; \lim_\beta \tau (b_\beta) \right) \\ &= \varphi (\sigma (a) \tau (b)) \\ &= \varphi (a \cdot b) \\ &= \varphi (a) \varphi (b) \qquad\qquad (a, b \in A) , \end{aligned}$$

which shows that $\varphi \in \Phi (A)$ and the proof is complete. □

Corollary 4.1. *Let A be a Banach algebra and σ, τ be two idempotent endomorphisms of A with dense range. If ${}_{\sigma}A_{\tau}$ is semisimple, then A is semisimple.*

Proof. Let $x \in rad\,(A)$. So for each $\varphi \in \Phi\,(A)$ we have $\varphi\,(x) = 0$. By the above proposition $\Phi\,({}_{\sigma}A_{\tau}) \subseteq \Phi\,(A)$ so,

$$\varphi\,(x) = 0 \qquad (\varphi \in \Phi\,({}_{\sigma}A_{\tau}))\,.$$

Thus $x \in rad\,({}_{\sigma}A_{\tau}) = \{0\}$ and so $x = 0$, which means that A is semisimple. □

It is easy to see that if A is $\star$-Banach algebra and σ is $\star$-idempotent endomorphism of A, i.e, an idempotent endomorphism such that $\sigma\,(a^*) = (\sigma\,(a))^*$, then ${}_{\sigma}A_{\sigma}$ is $\star$-Banach algebra. Also, it has proved that if A is a commutative semisimple Banach algebra, then every involution on A is continuous, see ([13], corollary 2.1.12). So we have the following result.

Corollary 4.2. *Let A be a commutative Banach algebra and σ be an idempotent endomorphism with dense range. If ${}_{\sigma}A_{\sigma}$ is semisimple, then every involution on A is continuous.*

Proposition 4.2. *Let A be a Banach algebra and I be a right (left) ideal in A. If $\sigma\,(I) \subseteq I$ $(\tau\,(I) \subseteq I)$, then I is a right (left) ideal in ${}_{\sigma}A_{\tau}$.*

Proof. For each $a \in A$ and $i \in I$ we have,

$$i \cdot a = \sigma\,(i)\,\tau\,(a) \in IA \subseteq I,$$

which shows that I is a right ideal in ${}_{\sigma}A_{\tau}$. □

Corollary 4.3. *Let A be a Banach algebra and I be a twosided ideal in A. If $\sigma\,(I) \subseteq I$ and $\tau\,(I) \subseteq I$, then I is a twosided ideal in ${}_{\sigma}A_{\tau}$.*

It has proved that if A is a contractible Banach algebra and I is a closed twosided ideal in A, then $\frac{A}{I}$ is contractible, [9]. So by proposition 10 we have the following result.

Corollary 4.4. *Suppose that A is a Banach algebra, σ is an idempotent endomorphism with dense range and τ is an idempotent epimorphism of A. Let I be a closed twosided ideal in A such that $\sigma\,(I) \subseteq I$ and $\tau\,(I) \subseteq I$. If any of the following conditions hold, then $\frac{{}_{\sigma}A_{\tau}}{I}$ is contractible.*

$i)$ A is τ-contractible.
$ii)$ A is σ-contractible .
$iii)$ A is (τ, σ)-contractible.
$iv)$ A is (σ, τ)-contractible.

5 Conclusion

By defining a new multiplication on Banach algebra A, we showed that the new Banach algebra ${}_{\sigma}A_{\tau}$, has better and stronger properties than Banach algebra A. For example, in Proposition 3.1, we showed that, if Banach algebra A is only σ-contractible, then the Banach algebra ${}_{\sigma}A_{\tau}$ is contractible, which is a stronger and better property than the σ-contractibility. Also, in Corollary 3.6, we showed that, if Banach algebra A be able to have σ-amenability property, then the Banach algebra ${}_{\sigma}A_{\tau}$ is amenable. Such results has been proven for more cases such as, approximate contractibility, approximate amenability, character contractibility and character amenability.

Competing Interests

The authors declare that no competing interests exist.

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