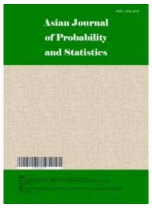

Asian Journal of Probability and Statistics

15(3): 35-57, 2021; Article no.AJPAS.75919
ISSN: 2582-0230

On Beta Exponentiated Moment Exponential Distribution with Mathematical Characteristics and Application to Engineering Sectors

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Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

Article Information

DOI: 10.9734/AJPAS/2021/v15i330356
Editor(s):
(1) Dr. Manuel Alberto M. Ferreira, Lisbon University, Portugal.
Reviewers:
(1) S. Nagesh, Karnatak University, India.
(2) Tahmineh Azizi, Florida State University, United States.
Complete Peer review History: https://www.sdiarticle4.com/review-history/75919

Original Research Article

Received 29 August 2021
Accepted 09 November 2021
Published 13 November 2021

Abstract

A new BEME distribution known as beta Exponentiated moment exponential (BEME) distribution is proposed. We provide here some shape properties, moments in the form of special functions, mean deviations of BEME distribution. We derive mathematical properties of the BEME distribution including the reliability measures, the Bonferroni and the Lorenz curves, rth order statistics, measures of uncertainty: the Shannon entropy measure and the s-entropy measure. The parameters of the BEME distribution are estimated by the method of maximum likelihood estimation and estimated non-linear equations for these estimates are presented. The application of BEME distribution is explored in three different fields of engineering.

Keywords: BEME distribution; EME distribution; Bonferroni curve; Lorenz curve; Renyi's entropy; s-entropy.

Subject Classification: 60E05, 46N30, 58Z05, 28D20, 62P30.

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1 Introduction

The class of Beta generalized distribution was first time presented by Eugene et al. [1] and they developed it through the logit of the beta random variable. They found the properties of the beta-normal distribution (BND) and showed a great flexibility in modeling. Jones [2] discussed a number of properties of beta generalized distributions.

Nadarajah and Kotz [3] suggested the beta-Gumbel distribution with a fact that it is a more tractable than the BND and found the moments in closed-form. The Beta Fréchet (BF) distribution was developed by Nadarajah and Gupta [4] by taking the baseline *cdf* of Frechet distribution in beta distribution and it is the generalization of the form of the Fréchet and exponentiated Fréchet (EF) distributions. Pescim et al. [5] proposed beta-generalized half normal distribution and derived expressions for the *cdf* and *pdf* which depends on simple functions. Barreto-Souza et al. [6] found the moments of BF distribution in terms of the moments of base line distribution.

The paper is organized as follows. Section 2 comprises of the general expansion of BEME'^s *pdf*, the hazard rates function with their graphical illustration, sub-models, and monotonicity of BEME distribution. Some structural properties of BEME distribution including moments' expressions with some numerical illustrations, reliability, the mean deviations about arithmetic mean and central value as median are described from Section 3 to Section 5. Further, in Section 6, we find the expressions for Bonferroni and Lorenz curves of BEME distribution. In Section 8, the method of maximum likelihood is used to estimate the unknown parameters of BEME distribution. We illustrate the application of this new distribution by applying it on three data sets. Finally, in section 9, some concluding remarks related to this distribution are given.

2 The BEME Distribution

In 2012, Dara [7] developed the moment exponential (ME) distribution and used it in reliability analysis. The *cdf* of ME distribution has the following expression

$$H_{ME}(x;\beta) = 1 - (1+\beta x)\, e^{-\beta x} \quad ; \quad x,\beta \in (0,\infty) \tag{1}$$

Hasnain et al. [8] introduced EME distribution as

$$G_{EME}(x;\alpha;\beta) = \left(H_{ME}(x;\beta)\right)^{\alpha} \quad ; \quad x,\alpha,\beta \in (0,\infty) \tag{2}$$

where $G_{EME}(x;\alpha;\beta)$ be the *cdf* of EME *r.v. X* and we define the *cdf* for BEME *r.v.* as

$$F(x) = I\left(G(x),a,b\right),; \quad x,a,b,\alpha,\beta \in (0,\infty) \tag{3}$$

where $I\left(x,m,n\right) = \dfrac{1}{B\left(m,n\right)}\displaystyle\int_0^x t^{m-1}\left(1-t\right)^{n-1}dt$ and $B\left(m,n\right) = \dfrac{\Gamma m\,\Gamma n}{\Gamma\left(m+n\right)}$.

(3) implies that the four-parameter BEME cdf is given by

$$F_{BEME}(x;\underline{\theta}) = \frac{1}{B(a,b)} \int_0^{G_{EME}(X,\alpha,\beta)} t^{a-1}(1-t)^{b-1}\,dt \tag{4}$$

for $x,a,b,\alpha,\beta \in (0,\infty)$, where $\underline{\theta} = (\alpha,\beta,a,b)$ and the BEME distribution *pdf* is given by

$$f_{BEME}\left(x;\ \underline{\theta}\right) = \frac{1}{B(a,b)}[G_{EME}(x)]^{a-1}[1-G_{EME(x)}]^{b-1}g_{EME}(x),$$

$$f_{BEME}\left(x;\ \underline{\theta}\right) = \frac{\alpha\beta^2}{B(a,b)}x\exp(-\beta x)\times\left(H_{ME}(x;\beta)\right)^{a\alpha-1}\times\left(1-\left(H_{ME}(x;\beta)\right)^{\alpha}\right)^{b-1} \tag{5}$$

for ; $x,a,b,\alpha,\beta \in (0,\infty)$

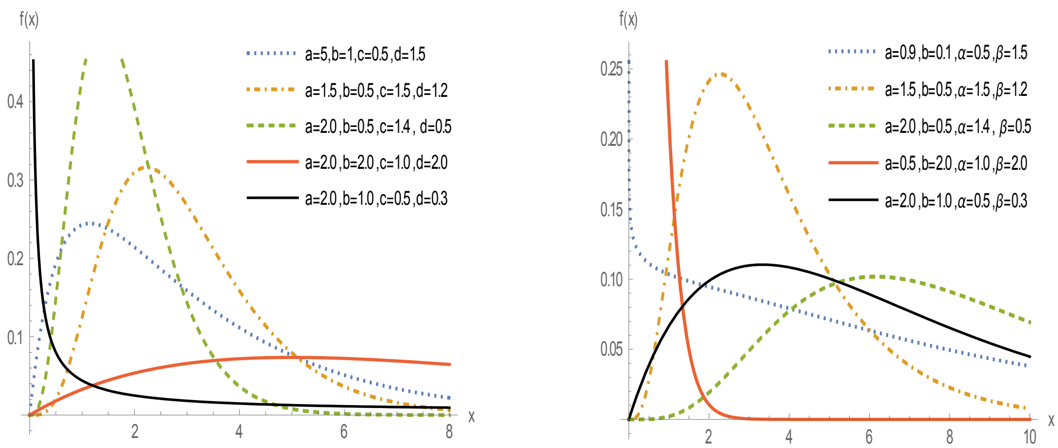

Fig. 1. *pdf* graphs of beta exponentiated moment exponential distribution

When product aα approach to zero the graph of function is reversed j-shaped. As aα = 1 distribution is unimodal. As aα >1 increase to infinity the distribution approaches positively skewed. When a ≤ 0.5 and α = β = b = 1 the *pdf* of distribution is reversed j-shaped and a > 0.5 the distribution is unimodal. For increasing the value of a > 0.5 the mode of BEME increases. For α > 1 and α approaches to ∞ then the BEME approach to symmetrical distribution.

2.1 Expansion of BEME density

We provide here the expansion for the probability density function of the BEME distribution depending if the parameter value is b > 0 is a real non-integer number or integer.

$$[1-\left(H_{ME}(x;\beta)\right)^{\alpha}]^{b-1} = \sum_{i=0}^{\infty}\tau_{i,b-1}\left(H_{ME}(x;\beta)\right)^{i\alpha} \qquad \text{where } \tau_{i,b-1} = (-1)^i\binom{b-1}{i}$$

The BEME density can be written in the following form

$$f_{BEME}(x,\underline{\theta}) = g_{GEME}(x)\sum_{i=0}^{\infty}\frac{\tau_{i,b-1}}{B(a,b)}\left(H_{ME}(x;\beta)\right)^{\alpha(a+i-1)}$$

$$= \alpha\beta^2 x\exp(-\beta x)\left(H_{ME}(x;\beta)\right)^{\alpha-1}\times\sum_{i=0}^{\infty}\frac{\tau_{i,b-1}}{B(a,b)}\left(H_{ME}(x;\beta)\right)^{\alpha(a+i-1)}$$

$$= \alpha\beta^2 x\exp(-\beta x)\times\sum_{i=0}^{\infty}\frac{\tau_{i,b-1}}{B(a,b)}\left(H_{ME}(x;\beta)\right)^{\alpha(a+i)-1}$$

$$f_{BEME}(x;\underline{\theta}) = \alpha\beta^2 x \exp(-\beta x)\sum_{i=0}^{\infty} l_i \left(H_{ME}(x;\underline{\theta})\right)^{(a+i)\alpha-1}$$

and the weights l_i are of the form $l_i = l_i(a,b) = \dfrac{\tau_{i,b-1}}{B(a,b)}$ such that $\sum_{i=0}^{\infty} l_i = 1$, for x > 0,

$[G_{EME}(x)]^{a+i-1}$ can be expanded as follows:

$$\begin{aligned}[G_{EME}(x)]^{a+i-1} &= \{1 - [1-G_{EME}(x)]\}^{a+i-1} \\ &= \sum_{j=0}^{\infty}(-1)\binom{a+i-1}{j}[1-G_{EME}(x)]^j ,\end{aligned}$$

with $[1-G_{EME}(x)]^j = \sum_{r=0}^{\infty}\tau_{r,j}\left(H_{ME}(x;\underline{\theta})\right)^r$,

$$[G_{EME}(x)]^{a+i-1} = \sum_{j=0}^{\infty}\ \sum_{r=0}^{\infty}(-1)^{r+j}\binom{a+i-1}{j}\binom{j}{r}[G_{EME}(x)]^r ,$$

The BEME density can be written in the form such as

$$\begin{aligned}f_{BEME}(x;\underline{\theta}) &= g_{EME}(x)\sum_{i,j=0}^{\infty}\sum_{r=0}^{j} l_{i,j,r}[G_{EME}(x)]^r \\ &= \alpha\beta^2 x\exp(-\beta x)\sum_{i,j=0}^{\infty}\sum_{r=0}^{j} l_{i,j,r}\left(H_{ME}\left(x,\beta\right)\right)^{(r+1)\alpha-1},\end{aligned}$$

where the weight $l_{i,j,r}$ is of the form

$$l_{i,j,r} = l_{i,j,r}(a,b) = \frac{(-1)^{i+j+r}}{B(a,b)}\binom{a+i-1}{j}\binom{b-1}{i}\binom{j}{r}$$

with $\sum_{i,j=0}^{\infty}\sum_{r=0}^{j} l_{i,j,r} = 1$, for $x,a,b,\alpha,\beta \in (0,\infty)$.

Clearly, the BEME density has the three different finite and infinite weighted power series sums of the baseline cdf of EME(x) for any real non-integer values of the parameters. After some rotational changing in limits we obtain

$$\begin{aligned}f_{BEME}(x;\underline{\theta}) &= g_{EME}(x)\sum_{i,r=0}^{\infty} p_i[G_{EME}(x)]^r \\ &= \alpha\beta^2 x\exp(-\beta x)\sum_{i,r=0}^{\infty} p_i\left(H_{ME}\left(x,\beta\right)\right)^{(r+1)\alpha-1} \qquad (6)\end{aligned}$$

where the weights p_i is of the form $p_i = p_i(a,b) = \dfrac{(-1)^i}{B(a,b)}\dbinom{b-1}{i} q_r\left(a+i-1\right)$

with $q_r = q_r(a+i-1) = \sum_{j=r}^{\infty}(-1)^{j+r}\dbinom{a+i-1}{j}\dbinom{j}{r}$, for $x,a,b,\alpha,\beta \in (0,\infty)$, respectively.

It can be seen that the BEME density is given in the form of weighted sums of the EME distribution function. Now, we deduce the sub-models of the BEME distribution by fixing parameters values.

(1) If the parameter value $\alpha = 1$, this is a new beta-ME (BME) distribution. The *cdf* of BME distribution is of the following form

$$F_{BME}(X;\Phi) = I\left(H_{ME}(x;\beta), a, b\right) \; where \; \Phi = \left(\beta, a, b\right) \in \left(0, \infty\right)$$

The BME *pdf* is given by

$$f_{BME}(x;\Phi) = \frac{\beta^2}{B(a,b)} x\left(1+\beta x\right)^{b-1} \exp(-b\beta x)\left(H_{ME}\left(x,\beta\right)\right)^{a-1}$$

(2) If $a = \alpha = 1$, then we have the *cdf* is of the form

$$F_{BME}(x;b,\beta) = b \int_0^{H_{ME}(x;\beta)} (1-t)^{b-1} dt$$

$$F_{BME}(x;\beta,b) = 1-\left(1+\beta x\right)^{b} \exp(-b\beta x) \text{ for } ; x, a, b, \beta \in \left(0, \infty\right).$$

and its pdf is $f_{BME}(x;\beta,b) = b\beta^2 x\left(1+\beta x\right)^{b-1} \exp(-b\beta x)$

(3) If $a = b = \alpha = 1$, then BEME *cdf* reduces to the ME *cdf* and its form is given by

$$F_{ME}(x;\beta) = 1-\left(1+\beta x\right)\exp(-\beta x) \text{ for } ; \quad x, \beta \in \left(0, \infty\right)$$

The ME's pdf has the following function

$f_{ME}(x;\beta) = \beta^2 x \exp(-\beta x)$ for ; $x, \beta \in \left(0, \infty\right)$, which is the ME *pdf.*

2.2 Reliability functions

We introduce the hazard and reverse hazard functions of the BEME distribution with graphical illustrations.

$$\begin{aligned} h_{BEME}(x;\underline{\theta}) &= \frac{f_{BEME}(x;\underline{\theta})}{\bar{F}_{BEME}(x;\underline{\theta})} \\ &= \frac{g_{BEME}(x)[G_{EME}(x)]^{a-1}[1-G_{EME}(x)]^{b-1}}{B(a,b)-B_{GEME(x)}(a,b)} \end{aligned} \tag{7}$$

$$\begin{aligned} r_{BEME}(x;\underline{\theta}) &= \frac{f_{BEME}(x;\underline{\theta})}{F_{BEME}(x;\underline{\theta})} \\ &= \frac{g_{BEME}(x)[G_{EME}(x)]^{a-1}[1-G_{EME}(x)]^{b-1}}{B_{GEME(x)}(a,b)} \end{aligned} \tag{8}$$

for $x > 0$,and for positive parameters' values. The function $G_{EME}(x)$ is defined in (2) and $g_{EME}(x)$ is the pdf of the (2).

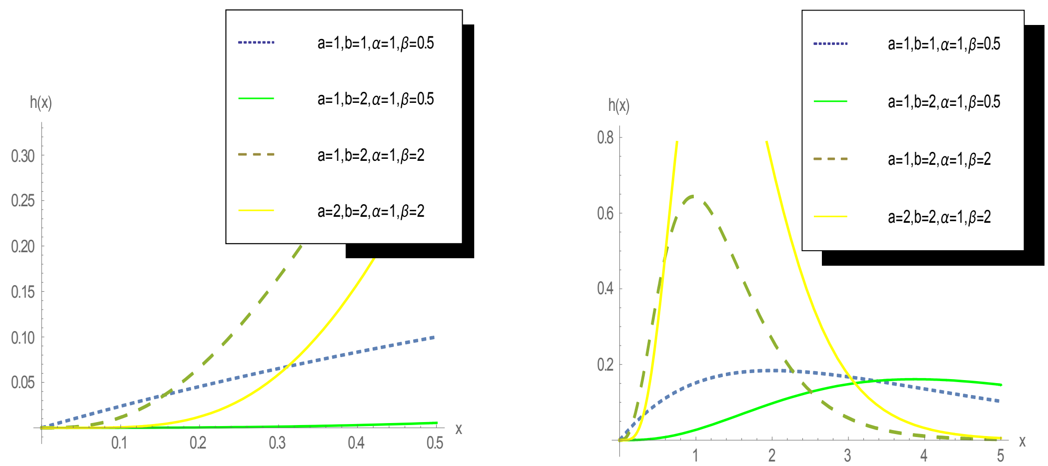

Fig. 2. Graph of *HRF* of BEME distribution

2.3 Monotonicity property

The monotonicity property of the BEME distribution.

$$\log f_{BEME}(x;\,\underline{\theta}) = \log\left(\frac{\alpha\beta^2}{B(a,b)}\right) + 1\log(x) + (b-1)\log[1-H^{\alpha}_{ME}(X;\beta)] - \beta x + (a\alpha-1)\log[H_{ME}(X;\beta)]$$

and

$$\frac{d\log f_{BEME}(x)}{dx} = \frac{1}{x} - \beta + \frac{a\alpha-1+[1+(1-b-a)\alpha H^{\alpha}{}_{ME}(X;\beta)}{H_{ME}(X;\beta)[1-H^{\alpha}{}_{ME}(X;\beta)]} H'_{ME}(X;\beta)$$

with $H'_{ME}(X;\beta) = \mathrm{d}H_{ME}(X;\beta)/\mathrm{dx} = \beta^2 x \exp(-\beta x)$ in above , we have

$$\frac{d\log f_{BEME}(x)}{dx} = \frac{1}{x} - \beta + \beta^2 x \exp(-\beta x)\left\{\frac{a\alpha-1+[1+(1-b-a)\alpha H^{\alpha}{}_{ME}(X;\beta)}{V(x)[1-H^{\alpha}{}_{ME}(X;\beta)]}\right\}$$

And

$$\frac{d^2\log f_{BEME}(x)}{dx^2} = -\frac{1}{x^2} + \frac{\beta^2}{H^2{}_{ME}(X;\beta)[1-H^{\alpha}{}_{ME}(X;\beta)]^2}\exp(-\beta x)q(x)$$

So that the value of $q(x)$ is as

$$\begin{aligned} q(x) = {} & (-\beta+1-\beta x)[(1+(1-b-a)\alpha)H^{\alpha}{}_{ME}(X;\beta)+a\alpha-1]H_{ME}(X;\beta)[1-H^{\alpha}{}_{ME}(X;\beta)] \\ & +x^2\beta^2\exp(-\beta x)[1+(1+(1-b-a)\alpha)H^{\alpha}{}_{ME}(X;\beta)][\alpha-1]H^{\alpha}{}_{ME}(X;\beta) \\ & -x^2\beta^2\exp(-\beta x)(a\alpha-1)[1-(1+\alpha)H^{\alpha}{}_{ME}(X;\beta)]. \end{aligned} \tag{9}$$

Analysis: For all positive values of the parameters generate such that

$$H'_{ME}(X;\beta)=\frac{dH_{ME}(X;\beta)}{dx}=\beta^2 x\exp(-\beta x)>0,\forall;x,\beta\in(0,\infty).$$

If x tends to 0, then the value of $H_{ME}(X;\beta)$ also tends to 0.

If x tends to ∞, then $H_{ME}(X;\beta)$ is also move to 1.

Therefore, we have the function $H_{ME}(X;\beta)$ is increasing monotonically from the interval 0 to 1. This implies that the functions $H^{\alpha}{}_{ME}(X;\beta)$ and $1-H^{\alpha}{}_{ME}(X;\beta)$ are also contain values from interval 0 to 1. If $\beta \geq 1$, $a\alpha - 1 < 1$ and $(a + b - 1)\alpha > 1$, we deduce the following

$$\frac{d\log f_{BEME}(x)}{dx}=\frac{1}{x}-\beta+\{\frac{a\alpha-1+[1+(1-b-a)\alpha H^{\alpha}{}_{ME}(X;\beta)}{H_{ME}(X;\beta)[1-H^{\alpha}{}_{ME}(X;\beta)]}\}H'_{ME}(X;\beta) < 0,$$

because $1 - x\beta < 0$ and $[1+(1- b - a)\alpha]\, H^{\alpha}{}_{ME}(X;\beta) < 1 - a\alpha$ and $H'_{ME}(X;\beta)/H_{ME}(X;\beta)$ [1- $H^{\alpha}{}_{ME}(X;\beta)$] > 0.

In this case the function $f_{BEME}(x;\underline{\theta})$ is always monotonically decreasing function for all x.

If β < 1, the density function $f_{BEME}(x;\underline{\theta})$ could attain the maximum point value, the minimum point value or the point of inflection if the following conditions holds

$$\frac{d^2 f_{EME}(x)}{dx^2}<0\ ,\quad \frac{d^2 f_{EME}(x)}{dx^2}>0 \text{ or } \frac{d^2 f_{EME}(x)}{dx^2}=0$$

3 Moments of the BEME Distribution

We find here the moments expressions of the BEME distribution and express them into special function used by Nadarajah et al., (2011). This section also comprises of the moments expressions of sub-models of BEME distribution Some numerical study of the moments is also presented at the end of section. We prove here a Lemma 1 with the help of result provided by Nadarajah et al., (2011) in their published paper.

LEMMA 3.1 Let

$$K(\underline{\varsigma})=\int_0^{\infty} x^p x[H_{ME}(x;n)]^{m-1}\exp(-qx)dx \qquad \textit{where } \underline{\varsigma}=(m,n,p,q)$$

$$K(\underline{\varsigma})=\sum_{l=o}^{\infty}\sum_{k=0}^{l}\binom{m-1}{l}\binom{l}{k}\frac{(-1)^l n^k\Gamma(p+k+2)}{(nl+q)^{p+k+1}}$$

(1) If the value of m is a non-integer type, we have

$$K(\underline{\varsigma})=\sum_{i=o}^{\infty}\sum_{k=0}^{l}\binom{m-1}{l}\binom{l}{k}\frac{(-1)^l n^k\Gamma(p+k+1)}{(nl+q)^{p+k+1}}$$

(2) If the value of m is an integer type, we have

$$K(\underline{\varsigma}) = \sum_{i=o}^{m-1}\sum_{k=0}^{l}\binom{m-1}{l}\binom{l}{k}\frac{(-1)^l n^k \Gamma(p+k+1)}{(nl+q)^{p+k+1}}$$

Proof (1) When the m is of non- integer type number, then

$$[H_{ME}(x;n)]^{m-1} = \sum_{l=o}^{\infty}\binom{m-1}{l}(-1)^l \times [\left(1+nx\right)\exp(-nx)]^l$$

and $$K(\underline{\varsigma}) = \sum_{l=o}^{\infty}\binom{m-1}{l}(-1)^l \times \int_0^{\infty} x^p x(1+nx)^l \exp[-(nx+q)x]dx.$$

Furthermore, since the number l is of an integer type , so we have the series as

$$(1+nx)^l = \sum_{k=o}^{l}\binom{l}{k}(nx)^k = \sum_{k=o}^{l}\binom{l}{k}n^k x^k$$

and $$K(\underline{\varsigma}) = \sum_{l=o}^{\infty}\binom{m-1}{l}(-1)^l \sum_{k=0}^{l}\binom{l}{k}n^k \int_0^{\infty} x^p x^{k+1} \exp[-(nl+q)x]dx.$$

$$k(\underline{\varsigma}) = \sum_{l=0}^{\infty}\binom{m-1}{l}(-1)^l \sum_{k=0}^{l}\binom{l}{k}n^k \int_0^{\infty} x^{p+k+2-1} \exp[-(nl+q)x]dx$$

$$= \sum_{l=o}^{\infty}\sum_{k=0}^{l}\binom{m-1}{l}\binom{l}{k}\frac{(-1)^l n^k \Gamma(p+k+2)}{(nl+q)^{p+k+2}} \tag{10}$$

(2) When the value of m is of an integral type number, the counter l in (10) stops at number $m-1$, so that,

$$k(\underline{\varsigma}) = \sum_{k=0}^{l}\sum_{l=0}^{m-1}\binom{m-1}{l}\binom{l}{k}\frac{(-1)n^k \Gamma(p+k+2)}{(nl+q)^{p+k+1}} \tag{11}$$

The sth moment expression of the BEME distribution denoted as μ_s' is found from the equation

$$\mu_s' = \int_0^{\infty} x^s f_{BEME}(x;\underline{\theta})\ dx$$

When $b > 0$ is such that it is a real type non-integer number, and First, if the number a is of an integer type number, from (10), so we have the sth moment expression as

$$\mu_s' = \alpha\beta^2 \sum_{i=0}^{\infty} l_i \int_0^{\infty} x^s x \exp(-\beta x) \times [H_{ME}(x;\beta)]^{(a+i)\alpha-1} dx$$

Now, using the above Lemma 3.1 with values m = (a + i)α, n = β, p = s, q = β, we deduce the result as

$$\mu_s' = \alpha\beta^2 \sum_{i=0}^{\infty} l_i K\left((a+i)\alpha, \beta, s, q\right)$$

If the parameter α is of non-integer type number, then the value of the number (a + i)α is also non-integer type number, and

$$\mu_s' = \frac{\alpha}{B(a,b)} \sum_{i,l=0}^{\infty} \sum_{k=0}^{l} \binom{b-1}{j} \binom{(a+i)\alpha-1}{l} \binom{l}{k} \times \frac{(-1)^{i+l}\Gamma(s+k+2)}{\beta^{s-k-1}(1+l)^{s+k+2}} \tag{12}$$

If the value of α is of an integer type number, then the expression (a + i)α contains also an integer number, such that the counter l in (12) stops working at value (a + i)$\alpha - 1$.

Second, if the number a is of real non-integer type number, then from (10), we can find easily

$$\mu_s' = \alpha\beta^2 \sum_{i,j=0}^{\infty} \sum_{r=0}^{j} l_{i,r,j} \int_0^{\infty} x^s \exp(-\beta x) \times [\beta x \exp(1-(1+\beta x))]^{(r+1)\alpha-1} dx$$

Applying the above Lemma 1, and with parameters values m = (r + 1)α, n = β , $s = p$, $q = \beta$, we have the following

$$\mu_s' = \alpha\beta^2 \sum_{i,j=0}^{\infty} \sum_{r=0}^{j} l_{i,r,j} K((r+1)\alpha, \beta, s, \beta)$$

If the parameter α is non- integer type number, then the expression (r +1) α is also non- integer type, and

$$\mu_s' = \frac{\alpha}{B(a,b)} \sum_{i,l=0}^{\infty} \sum_{r=0}^{j} \sum_{k=0}^{l} \binom{b-1}{j} \binom{a+i-1}{j} \binom{b-1}{i} \binom{j}{r} \times \binom{(r+1)\alpha-1}{l} \binom{l}{k} \times \frac{(-1)^{i+j+r+l}\Gamma(s+k+1)}{\beta^{s-k-1}(1+l)^{s+k+1}} \tag{13}$$

If the value of parameter α is of an integer type, then the expression (r +1) α is also a number of integer type, such that the value of the counter l in (13) completes at value (r + 1)α – 1.

Table 1. Moments and related measures of the BEME distribution for fixed parameters' values *i.e* $(\alpha,\beta)=(0.5,1)$

μ_s'	$(a,b)=(1,1.5)$	$(a,b)=(1.5,1.5)$	$(a,b)=(2,2.5)$	$(a,b)=(2.5,2.5)$
μ_1'	0.301234	0.591582	0.741137	1.32273
μ_2'	0.710192	0.738389	0.883784	2.25288
μ_3'	1.845821	2.348553	2.124572	5.07719
μ_4'	2.413785	4.014563	5.342733	6.5798
σ^2	0.61945028	0.38842004	0.3344994	0.503259
S.K	2.10307249	2.28870354	1.17199915	0.226493
Kurtosis	18.36157	7.245834	5.577122	3.212824

By changing $\sum_{j=0}^{\infty}\sum_{r=0}^{j}$ to $\sum_{r=0}^{\infty}\sum_{j=r}^{\infty}$, in sth moment, we have

$$\mu_s' = \alpha\beta^2 \sum_{i,r=0}^{\infty} p_i \int_0^{\infty} x^s x \exp(-\beta x) \times [\beta x \exp(-\beta x)]^{(r+1)\alpha-1} dx .$$

Applying the Lemma 1 on above integral, with m =(r+1)α is a non- integer type number, then we have the sth moment expression as

$$\mu_s' = \frac{\alpha}{B(a,b)} \sum_{i,r,l=0}^{\infty} \sum_{j=r}^{\infty} \sum_{k=0}^{l} \binom{a+j-1}{j} \binom{b-1}{i} \binom{r}{j} \times \binom{(r+1)\alpha-1}{l} \binom{l}{k} \frac{(-1)^{i+j+r+l}\Gamma(s+k+2)}{\beta^{s+w+k-1}(1+l)^{s+k+2}}$$

4 Reliability

In this section, we derive the expression of reliability R when the random variables X and Y have independent BEME distributions and their set of parameters are $\underline{\theta}_1 = (\alpha_1, \beta_1, a_1, b_1)$ and $\underline{\theta}_2 = (\alpha_2, \beta_2, a_2, b_2)$ distributions, respectively. And we use the (2) so that

$$\begin{aligned} F_{BEME}(x;\underline{\theta}) &= \frac{1}{B(a,b)} \int_0^{G_{EME}(X,\alpha,\beta)} t^{a-1}(1-t)^{b-1} dt \\ &= \frac{1}{B(a,b)} \sum_{j=0}^{\infty} \tau_{j,b-1} [H_{ME}(x;\beta)]^{(a+j)\alpha} \end{aligned}$$

So the reliability measures R, we obtain its expression from the following formula

$$\begin{aligned} R &= P(X > Y) \\ &= \int_0^{\infty} f_x(x;\underline{\theta}_1) F_y(x;\underline{\theta}_2) dx \\ &= \int_0^{\infty} \frac{\alpha_1 \beta_1^2}{B(a_1,b_1)} x \exp(-\beta_1 x) \times [H(x,\beta_1)]^{a_1\alpha_1-1} \\ &\qquad \times [1 - H^{\alpha_1}(x,\beta_1)]^{b-1} \times \frac{1}{B(a_2,b_2)(a_2+j)} \times \sum_{j=0}^{\infty} \tau_{j,b_2-1} [H(x,\beta_2)]^{(a_a+j)\alpha_2} dx. \end{aligned}$$

We apply the following series representations:

$$\begin{aligned} [H(x,\beta_1)]^{a_1\alpha_1-1} &= \sum_{k=0}^{\infty} \binom{\alpha_1\alpha_2-1}{k} (-1)^k [(1+\beta_1 x)\exp(-\beta_1 x)]^k \\ &= \sum_{k=0}^{0} \binom{\alpha_1\alpha_2-1}{k} (-1)^k \sum_{m=0}^{k} \binom{m}{k} \beta^m{}_1 x^m \exp(-k\beta_1 x)] \\ &= \sum_{l,p=0}^{\infty} \sum_{n=0}^{p} \binom{b_1-1}{l} \binom{\alpha_1 l}{p} \binom{p}{n} (-1)^{l+p} \beta^n{}_1 x^n \exp(-\beta_1 p x) \end{aligned}$$

$$\left(1-[H\left(x,\beta_1\right)]^{\alpha_1}\right)^{b_1-1}=\sum_{l,p=0}^{\infty}\sum_{n=0}^{p}\binom{b_1-1}{l}\binom{\alpha_1 l}{p}\binom{p}{n}(-1)^{l+p}\,\beta_1{}^n x^n\exp\left(-\beta_1 px\right)$$

and

$$[H\left(x,\beta_2\right)]^{(a_2+j)\alpha_2}=\sum_{q=0}^{\infty}\sum_{t=0}^{q}\binom{(a_2+j)\alpha_2}{q}\binom{q}{t}(-1)^q\,\beta^t{}_2x^t\exp(-\beta_2 qx)$$

After simplifying we have the value of R

$$\begin{aligned}R=\int_0^{\infty}\frac{\alpha_1\beta_1^2}{B\left(a_1,b_1\right)}x\exp(-\beta_1 x)\times\sum_{k=0}^{\infty}\sum_{m=0}^{k}\binom{a_1\alpha_1-1}{k}\binom{k}{m}(-1)^k\,\beta^m{}_1x^m\exp(-\beta_1 kx)\\ \times\sum_{l,p=0}^{\infty}\sum_{n=0}^{p}\binom{b_1-1}{l}\binom{\alpha_1 l}{p}\binom{p}{n}(-1)^{l+p}\,\beta^n{}_1x^n\exp(-\beta_1 px)\\ \times\frac{1}{B(a_2,b_2)(a_2+j)}\sum_{j=0}^{\infty}\binom{b_2-1}{j}(-1)^j\\ \times\sum_{q=0}^{\infty}\sum_{t=0}^{q}\binom{(a_2+j)\alpha_2}{q}\binom{q}{t}(-1)^q\,\beta^t{}_2x^t\exp(-\beta_2 qx)dx\\ =C\sum_{k,l,p,j,q=0}^{\infty}\sum_{m=0}^{k}\sum_{n=0}^{p}\sum_{t=0}^{q}\binom{(a_1\alpha_1-1)}{k}\binom{k}{m}\binom{b_1-1}{l}\binom{\alpha_1 l}{p}\\ \times\binom{p}{n}\binom{b_2-1}{j}\binom{(a_2+j)\alpha_2}{q}\binom{q}{t}\frac{(-1)^{k+l+p+j+q}\beta_1^{m+n+2}\beta_2^{\,t}}{(a_2+j)}\\ \times\int_0^{\infty}xx^{m+n+t}\exp(-[\beta_1(1+k+p)+\beta_2 q]x)dx.\end{aligned}$$

where $C=\dfrac{\alpha_1}{B(a_1,b_1)B(a_2,b_2)}$

After applying the basic definition of the usual gamma function, we have the result as that

$$\int_0^{\infty}x^{m+n+t+1}\exp(-[\beta_1(1+k+p)+\beta_2 q]x)dx=\frac{(m+n+t+1)!}{(\beta_1(1+k+p)+\beta_2 q)^{m+n+t+2}}$$

After replacing this result into R we obtain

$$\begin{aligned}R=C\sum_{k,l,p,j,q=0}^{\infty}\sum_{m=0}^{k}\sum_{n=0}^{p}\sum_{t=0}^{q}\binom{(a_1\alpha_1-1)}{k}\binom{k}{m}\binom{b_1-1}{l}\binom{\alpha_1 l}{p}\\ \times\binom{p}{n}\binom{b_2-1}{j}\binom{(a_2+j)\alpha_2}{q}\binom{q}{t}\frac{(-1)^{k+l+p+j+q}\beta_1^{m+n+2}\beta_2^{\,t}}{(a_2+j)}\\ \times\frac{(m+n+t+1)!}{(\beta_1(1+k+p)+\beta_2 q)^{m+n+t+2}}\end{aligned}$$

Secondly, we obtain

$$R=\int_0^\infty \frac{\alpha_1\beta_1^2}{B(a_1,b_1)} x\exp(-\beta_1 x)\times\sum_{k=0}^{\infty}\sum_{m=0}^{k}\binom{a_1\alpha_1-1}{k}\binom{k}{m}(-1)^k \beta^m{}_1 x^m \exp(-\beta_1 kx)$$

$$\times\sum_{l,p=0}^{\infty}\sum_{n=0}^{p}\binom{b_1-1}{l}\binom{\alpha_1 l}{p}\binom{p}{n}(-1)^{l+p}\beta^n{}_1 x^n \exp(-\beta_1 px)$$

$$\times\frac{1}{B(a_2,b_2)(a_2+j)}\sum_{j=0}^{\infty}\binom{b_2-1}{j}(-1)^j[\beta_2 x\exp(-\beta_2 x)]^{(a_2+j)\alpha_2}dx$$

$$=\frac{\alpha_1}{B(a_1,b_1)B(a_2,b_2)}\sum_{k,l,p,j,q=0}^{\infty}\sum_{m=0}^{k}\sum_{n=0}^{p}\binom{a_1\alpha_1-1}{k}\binom{k}{m}\binom{b_1-1}{l}\binom{\alpha_1 l}{p}$$

$$\times\binom{p}{n}\binom{b_2-1}{j}\binom{(a_2+j)\alpha_2}{q}\frac{(-1)^{k+l+p+j}\beta_1^{m+n+2}}{(a_2+j)}$$

$$\times\int_0^\infty xx^{+n}e^{-\beta_1(1+k+p)x}[\beta_2 xe^{-\beta_2 x}]^{(a_2+j)\alpha_2}dx$$

$$=C\sum_{k,l,p,j,q=0}^{\infty}\sum_{m=0}^{k}\sum_{n=0}^{p}\binom{a_1\alpha_1-1}{k}\binom{k}{m}\binom{b_1-1}{l}\binom{\alpha_1 l}{p}$$

$$\times\binom{p}{n}\binom{b_2-1}{j}\frac{(-1)^{k+l+p+j}\beta_1^{m+n+2}}{(a_2+j)}\times k((a_2+j)\alpha_2+1,\beta_2,m+n,\beta_1(1+k+p)).$$

where $C=\dfrac{\alpha_1}{B(a_1,b_1)B(a_2,b_2)}$

5 Mean Deviations of BEME Distribution

The expressions of mean deviations about mean (μ = E(X)) and about the median (M = Median(X)) are defined by

$$\delta_1(X)=\int_0^\infty |x-\mu|\,f(x)dx,\ and\ \delta_2(X)=\int_0^\infty |x-M|\,f(x)dx$$

The expressions of $\delta_1(X)$ and $\delta_2(X)$ can be found and results are presented here:

$$\delta_1(X)=2\mu F(\mu)-2\mu+2\int_\mu^\infty xf(x)dx$$

$$\delta_2(X)=-\mu+2\int_M^\infty xf(x)dx$$

respectively. And by using the expressions of arithmetic mean for BEME distribution and of the lower truncated arithmetic mean are

$$\mu = \int_0^\infty x f_{BEME}(x)dx$$

$$= \alpha\beta^2 \sum_{i,j=0}^{\infty} \sum_{r=0}^{j} l_{i,j,r} K\left((r+1)\alpha, \beta, 1, \beta\right)$$

and

$$\int_\mu^\infty x f_{BEME}(x)dx = \alpha\beta^2 \sum_{i,j=0}^{\infty} \sum_{r=0}^{j} l_{i,j,r} L((r+1)\alpha, \beta, 1, \beta, \mu)$$

and

$$\int_M^\infty x f_{BEME}(x)dx = \alpha\beta^2 \sum_{i,j=0}^{\infty} \sum_{r=0}^{j} l_{i,j,r} L((r+1)\alpha, \beta, 1, \beta, M)$$

So that

$$\delta_1(X) = 2\mu F_{BEME}(\mu) - 2\mu \frac{\alpha\beta^2}{B(a,b)} \sum_{i,j=0}^{\infty} \sum_{r=0}^{j} l_{i,j,r} L((r+1)\alpha, \beta, 1, \beta, \mu)$$

and

$$\delta_2(X) = -\mu + 2\alpha\beta^2 \sum_{i,j=0}^{\infty} \sum_{r=0}^{j} l_{i,j,r} L((r+1)\alpha, \beta, 1, \beta, M)$$

6 Bonferroni and Lorenz curves of BEME distribution

The Bonferroni and the Lorenz curves are usually based on incomplete moments and are defined in the following expressions as

$$B(P) = \frac{1}{p}\int_0^q x f(x)dx, \qquad L(P) = \frac{1}{\mu}\int_0^q x f(x)dx$$

respectively, where μ is the mean of BEME distribution and q is the inverse percentile function. Now, the expressions of the Bonferroni and the Lorenz curves for the BEME distribution respectively are as under:

$$B(p) = \frac{1}{p} - \frac{\alpha\beta^2}{p\mu} \sum_{i,j=0}^{\infty} \sum_{r=0}^{j} l_{i,j,r} L\left((r+1)\alpha, \beta, 1, \beta, q\right)$$

and $$L(p) = 1 - \frac{\alpha\beta^2}{\mu} \sum_{i,j=0}^{\infty} \sum_{r=0}^{j} l_{i,j,r} L\left((r+1)\alpha, \beta, 1, \beta, q\right)$$

respectively.

7 Order Statistics and Different Measures of Uncertainty

The following section comprises the distribution of kth order statistics and measures of uncertainty such as Renyi's Entropy and s-Entropy for the BEME distribution are presented. In information theory we usually use the concept of entropy when we have to measure the uncertainty and its value will be maximum when the outcome of the random variable half chance of appearing.

7.1 Distribution of order statistics

Suppose that random sample of size n are taken from BEME distribution and the related order statistics are arranged increasing with usual notations then probability density function of this kth order statistics, say $Y_K = X_{k;n}$ is derived as under.

$$f_k(y_k)=\frac{n!}{(k-1)!(n-k)!}\sum_{l=0}^{n-k}\binom{n-k}{l}(-1)^l\left(\frac{B_{EME(y_{k;\alpha,\beta})}(a,b)}{B(a,b)}\right)^{k-1+l}$$
$$\times\frac{\alpha\beta^2}{B(a,b)}y_k\exp(-\beta y_k)[H(y_k)]^{a\alpha-1}[1-H^{\alpha}(y_k)]^{b-1}$$
$$=\frac{\alpha\beta^2 n!\,y_k\exp(-\beta y_k)}{B(a,b)^{k+1}(k-1)!(n-k)!}\sum_{l=0}^{n-k}\sum_{m=0}^{b-1}\binom{n-k}{l}\binom{b-1}{m}(-1)^{l+m}$$
$$\times(B_{EME(y_{k;\alpha,\beta})}(a,b))^{k-1+l}[H(y_k)]^{(a+m)\alpha-1},$$

Where $H(y_k) = 1 - (1+\beta y_k)/\exp(-\beta y_k)$ and $G_{EME}(y_k;\alpha,\beta) = H^{\alpha}(y_k)$.

The *cdf* of this order statistics y_k is written as

$$F_K(y_k)=\sum_{j=k}^{n}\sum_{l=0}^{n-j}\binom{n}{j}\tau_{l,n-j}[F(y_k)]^{j+l}$$
$$=\sum_{j=k}^{n}\sum_{l=0}^{n-j}\binom{n}{j}/[B(a,b)]^{j+l}\ \tau_{l,n-j}(B_{EME(y_k\alpha,\beta)}(a,b))^{j+l}$$

7.2 Renyi's entropy

Renyi entropy [9] (RE) is the general form of the Shannon entropy [10]. The RE is defined in the following form of measurement

$$H_{\gamma}(f_{BEME}(x;\underline{\theta}))=\frac{1}{1-\gamma}\log\int_0^{\infty}f^{\gamma}{}_{BEME}(x;\underline{\theta})\,dx$$

where $\gamma > 0$ and $\gamma \neq 1$. Moreover, Renyi entropy provides the value of Shannon entropy when $\gamma \to 1$. Now

$$\int_0^{\infty}f^{\gamma}{}_{BEME}(x;\underline{\theta})dx=\left(\frac{\alpha\beta^2}{B(a,b)}\right)^{\gamma}\int_0^{\infty}x^{\gamma}\exp(-\beta\gamma x)[H(x)]^{\alpha a\gamma-\gamma}[1-H^{\alpha}(x)]^{b\gamma-\gamma}dx.$$

Note that

$$[H(x)]^{\alpha a\gamma-\gamma}=[\beta x\exp(-\beta x)]^{\alpha a\gamma-\gamma}$$
$$=\sum_{k=0}^{\infty}(-1)^k\binom{\alpha a\gamma-\gamma}{k}\sum_{j=0}^{k}\binom{k}{j}\beta^j x\exp(-\beta kx)$$

And

$$[1-H^{\alpha}(x)]^{b\gamma-\gamma}=\sum_{m=0}^{\infty}(-1)^m\binom{b\gamma-\gamma}{m}\times\sum_{n=0}^{\infty}(-1)^n\binom{\alpha m}{n}\sum_{t=o}^{n}\binom{n}{t}\beta^t x\exp(-\beta nx)$$

After substituting values in definaion we obtain

$$\int_0^\infty f^\gamma{}_{BEME}(x;\underline{\theta})dx = \left(\frac{\alpha\beta^2}{B(a,b)}\right)^\gamma \sum_{k=0}^\infty \sum_{j=0}^k \sum_{m=0}^\infty \sum_{n=0}^\infty \sum_{t=0}^n \binom{\alpha\alpha\gamma-\gamma}{k}\binom{k}{j}\binom{b\gamma-\gamma}{m}\binom{\alpha m}{n}\binom{n}{t}$$

$$\times(-1)^{k+m+n\beta j+t}\int_0^\infty x^{\gamma+j+t}\exp(-\beta(\gamma+k+n)x)dx$$

$$=\left(\frac{\alpha\beta^2}{B(a,b)}\right)^\gamma \times \sum_{k,m,n}^\infty \sum_{j=0}^k \sum_{t=0}^n \binom{\alpha\alpha\gamma-\gamma}{k}\binom{k}{j}\binom{b\gamma-\gamma}{m}\binom{\alpha m}{n}\binom{n}{t}$$

$$\times\frac{(-1)^{k+m+n}e^{\beta(\gamma+K+n)}\Gamma(\gamma+j+t+1)}{(\gamma+k+n)^{\gamma+j+t+1\beta\gamma+1}}$$

And finally the Renyi entropy for BEME distribution is

$$H_\gamma(f_{BEME}(x;\underline{\theta})) = \frac{1}{1-\gamma}\log\left(\int_0^\infty f^\gamma{}_{BEME}(x;\alpha,\beta,a,b)dx\right)$$

$$=\frac{\gamma}{1-\gamma}\log\left(\frac{\alpha\beta^2}{B(a,b)}\right) + \frac{1}{1-\gamma}\log\left\{\sum_{k,m,n}^\infty \sum_{j=0}^k \sum_{t=0}^n \binom{\alpha\gamma-\gamma}{k}\binom{k}{j}\binom{b\gamma-\gamma}{m}\binom{\alpha m}{n}\binom{n}{t}\right\}$$

$$\times\frac{(-1)^{k+m+n}e^{\beta(\gamma+K+n)}\Gamma(\gamma+j+t+1)}{(\gamma+k+n)^{\gamma+j+t+1\beta\gamma+1}}$$

For $\gamma = 1$, we obtain the expression of Shannon entropy for BEME distribution which is as follows

$$E[-\log f_{BEME}(X;\underline{\theta})] = -\log\left(\frac{\alpha\beta^2}{B(a,b)}\right) - E[\log x] + \beta E(X)$$

$$+(1-\alpha\alpha)E[\log H(X)] + (1-b)E[\log(1-H^\alpha(X))]$$

Note that,

$$\log(1-H^\alpha(X) = -\sum_{k=1}^\infty [H(x)]^{\alpha k}/k, and$$

$$E[\log(1-H^\alpha(x))] = -\sum_{k=1}^\infty \frac{1}{k}E[H(x)]^{\alpha k}$$

Also

$$\log H(x) = -\sum_{k=1}^\infty \sum_{l=0}^k \binom{k}{l}\frac{\beta^l}{k}x^l\exp(-\beta kx),$$

So that

$$E[\log H(x)] = -\sum_{k=1}^\infty \sum_{l=0}^k \binom{k}{l}\frac{\beta^l}{k}E[X^l\exp(-\beta kx)],$$

By using the expansion of density function, When the values of a and b are real and non-integer type, then we have the expression of Shannon entropy as

$$E[\log H(X)] = -\frac{\alpha}{B(a,b)}\sum_{k=1}^\infty \sum_{l=0}^l \sum_{i,j=0}^\infty \sum_{r=0}^j \binom{k}{l}\binom{a+i-1}{j}\binom{b-1}{i}\binom{j}{r}$$

$$\times(-1)^{i+j+r}\beta^{l+2}k((r+1)\alpha,\beta,l,\beta(1+k))$$

and

$$E[\log(1-H^{\alpha}(X)] = -\frac{\alpha}{B(a,b)}\sum_{k=1}^{\infty}\sum_{i.j=0}^{\infty}\sum_{r=0}^{j}\binom{a+i-1}{j}\binom{b-1}{i}\binom{j}{r}$$
$$\times(-1)^{i+j+r}\beta^{l+2}k((r+k+1)\alpha,\beta,0,\beta).$$

7.3 s – Entropy

Another general form of entropy measure is the s-entropy for the BEME distribution is defined by

$$H_s(f_{BEME}(x;\underline{\theta})) = \begin{cases} \dfrac{1}{s-1}[1-\int_0^{\infty} f^{s}{}_{BEME}(x;\underline{\theta})dx] \; if \; s>0 \\ \dfrac{1}{s-1}[1-E[-\log f_{BEME}(x;\underline{\theta})] \; if \; s=1 \end{cases}$$

After some simplification we have

$$=\left(\frac{\alpha^{s}\beta^{s-1}}{[B(a,b)]^{s}}\right)^{\gamma}\times\sum_{k,m,n=0}^{\infty}\sum_{j=0}^{k}\sum_{t=0}^{n}\binom{\alpha s-s}{k}\binom{k}{j}\binom{bs-s}{m}\binom{\alpha m}{n}\binom{n}{t}$$
$$\times\frac{(-1)^{k+m+n}e^{\beta(s+K+n)}\Gamma(s+j+t+1,\beta(s+k+n))}{(s+k+n)^{s+j+t+1}}$$

Finally, we obtain for s ≠ 1 and s > 0

$$H_s\left(f_{BEME}(x;\underline{\theta})\right) = \frac{1}{1-s}-\left(\frac{\alpha^{s}\beta^{s-1}}{(s-1)[B(a,b)]^{s}}\right)^{\gamma}\times\sum_{k,m,n=0}^{\infty}\sum_{j=0}^{k}\sum_{t=0}^{n}\binom{\alpha s-s}{k}\binom{k}{j}\binom{bs-s}{m}\binom{\alpha m}{n}\binom{n}{t}$$
$$\times\frac{(-1)^{k+m+n}e^{\beta(s+K+n)}\Gamma(s+j+t+1,\beta(s+k+n))}{(s+k+n)^{s+j+t+1}}$$

8 Maximum Likelihood Estimation

This section contains the expressions of maximum likelihood expressions and its applications to some real data sets which shows the applicability of this new model on various fields. If a random sample of size n is taken as $x_1,\ldots,x_n$ from BEME distribution, we obtain the likelihood function as follows:

$$L(\underline{\theta}) = \left[\frac{\alpha\beta^{2}}{B(a,b)}\right]^{n}\left(\prod_{i=1}^{n}x\right)\exp(-\beta\sum_{i=0}^{n}x_i)\times\prod_{i=1}^{n}\left(H_{ME}(x;\beta)\right)^{a\alpha-1}\times\prod_{i=1}^{n}\left(1-\left(H_{ME}(x;\beta)\right)^{\alpha}\right)^{b-1}$$

The log-likelihood function is given by

$$\log L(\underline{\theta}) = n\log\left[\frac{\alpha\beta^{2}}{B(a,b)}\right]+\prod_{i=1}^{n}\log x\exp(-\beta\sum_{i=0}^{n}x_i)$$
$$+(\alpha\alpha-1)\prod_{i=1}^{n}\log H(x_i)+(b-1)\sum_{i=1}^{n}\log[1-H^{\alpha}(x_i)] \tag{14}$$

The partial derivatives of $\log L(\underline{\theta})$ are

$$\frac{\partial \log L(\underline{\theta})}{\partial a} = n\left[\psi(a+b)-\psi(a)\right] + a\sum_{i=1}^{n}\log H(x_i)$$

$$\frac{\partial \log L(\underline{\theta})}{\partial b} = n\left[\psi(a+b)-\psi(b)\right] + \sum_{i=1}^{n}\log[1-H^{\alpha}(x_i)]$$

$$\frac{\partial \log L(\underline{\theta})}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}\log H(x_i) + (1-b)\sum_{i=1}^{n}\frac{H^{\alpha}(x_i)\log H(x_i)}{1-H^{\alpha}(x_i)}$$

And

$$\frac{\partial \log L(\underline{\theta})}{\partial \beta} = n\frac{B(a,b)}{\alpha\beta^2}\frac{\partial \alpha\beta^2 / B(a,b)}{\partial \beta}$$

$$-\sum_{i=1}^{n}x_i + (\alpha\alpha-1)\sum_{i=1}^{n}\frac{\partial H(x_i)/\partial\beta}{H(x_i)} + \alpha(1-b)\sum_{i=1}^{n}\frac{[H(x_i)]^{\alpha-1}\partial H(x_i)/\partial\beta}{1-H^{\alpha}(x_i)}$$

$$\frac{\partial(\alpha\beta^2 / B(a,b)}{\partial\beta} = \frac{2\alpha\beta}{B(a,b)}, \quad \frac{\partial H(x_i)}{\partial\beta} = \beta x_i \exp(-\beta x_i)$$

$$\frac{\partial \log L(\underline{\theta})}{\partial \beta} = \frac{n(2+\beta)}{\beta(1+\beta)} - \sum_{i=1}^{n}x_i + \sum_{i=1}^{n}\beta(2+\beta+x_i+\beta x_i)x_i e^{-\beta x_i} \times\left[\frac{\alpha\alpha-1}{H(x_i)} + \frac{\alpha(1-b)[H(x_i)]^{\alpha-1}}{1-H^{\alpha}(x_i)}\right]$$

Since all the expressions of derivatives are non linear therefore to find numerical estimates of the parameters can be found through Newton- Raphson procedure. We can use here also computer techniques such as MAPLE or MATLAB to find the solution set of equations $(\partial\log L/\partial\alpha, \partial\log L/\partial\beta, \partial\log L/\partial a, \partial\log L/\partial b)^T = 0$

8.1 Application on real data

In this section, we report the flexibility and potentiality the BEME distribution in modeling real data from engineering sciences. For this, we consider three suitable lifetime data sets and find numerical estimates of the parameters with their standard errors. We compare here the BEME distribution with its own sub-models and some other competitor's models. The R Software version 3.3.4 are used to find the following tables and graphs. Tables 2, 3, and 4, show the estimates and their standard errors (in parenthesis) as well as the goodness-of-fit criterions of the proposed BEME model and competing models, for the three data sets, respectively. Furthermore tables contain - maximized log-likelihood (-LL), and Kolmogorov Smirnov test (KS).

Eventually, the performance of the BEME fulfills the criteria of a better fit based on the results in Tables 2-4. Consequently, we declare that the BEME distribution provides a better fit among all competing models for the three lifetime data sets.

Moreover, the plots of fitted PDF (Figs. 3, 6, 9), CDF (Figs. 4, 7, 10) and Q-Q Plot (Figs. 5, 8, 11) of the BEME distribution for the three data sets are presented in Figs. 9, 10 and 11, respectively. These plots reveal that the BEME distribution provides close fits to the three real datasets.

First Data Set: The following censored (in Gba) values about the breaking stress of carbon fibers discussed by Nicholas and Padgett [11], present almost the symmetric trend of data and the values are: 3.70, 2.74, 2.73, 2.50, 3.60, 3.11, 3.27, 2.87, 1.47, 3.11, 3.56, 4.42, 2.41, 3.19, 3.22, 1.69, 3.28, 3.09, 1.87, 3.15, 4.90, 1.57, 2.67, 2.93, 3.22, 3.39, 2.81, 4.20, 3.33, 2.55, 3.31, 3.31, 2.85, 1.25, 4.38, 1.84, 0.39, 3.68, 2.48, 0.85, 1.61, 2.79, 4.70, 2.03, 1.89, 2.88, 2.82, 2.05, 3.65, 3.75, 2.43, 2.95, 2.97, 3.39, 2.96, 2.35, 2.55, 2.59, 2.03, 1.61, 2.12, 3.15, 1.08, 2.56, 1.80, 2.53.

Table 2. Parameters estimates, log-likelihood

Data set size	Model	α	β	A	B	-2ln(L)	k-s value
	BEME (α,β,a,b)	2.4568 (1.6317)	0.154 (0.070)	0.567 (3.4×10^{-1})	407.5 (1.68×10^{-4})	168.4	0.04
	GL	7.0411 (1.673)	1.2461 (0.109)	1	2922.5 (8.9×10^{-6})	187.6	0.15
n=66	BL	1	0.590 (0.053)	1	1	244.8	0.18
	BE	4.070 (1.273)	0.092 (0.022)	0.759 (0.368)	120 (7.09×10^{-6})	171.8	0.20
	Weibull	3.441 (0.330)	0.326 (0.0122)	1	1	172.1	0.20
	L	1	0.023 (4.6×10^{-3})	7.509 (1.279)	111.3 (8.04×10^{-6})	182.3	0.24
	EE	3.1524 (0.152)	0.1561 (0.112)	--	--	190.04	0.14
	EME (α,β,1,1)	4.2139 (0.183)	0.273 (0.061)	--	--	188.12	0,13

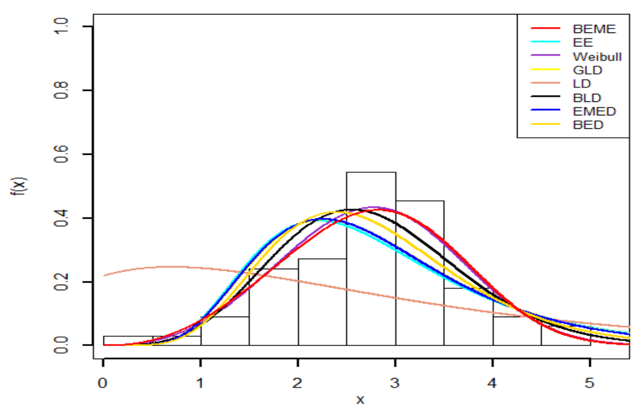

Fig. 3. PDF graphs for first data set

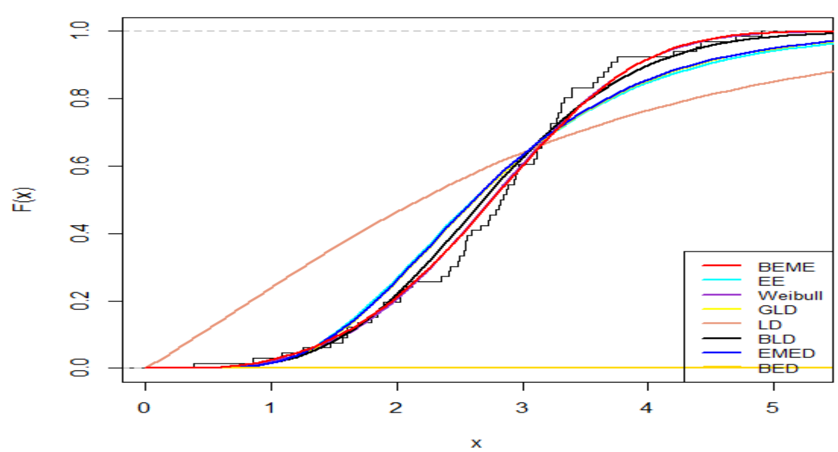

Fig. 4. CDF graphs for first data set

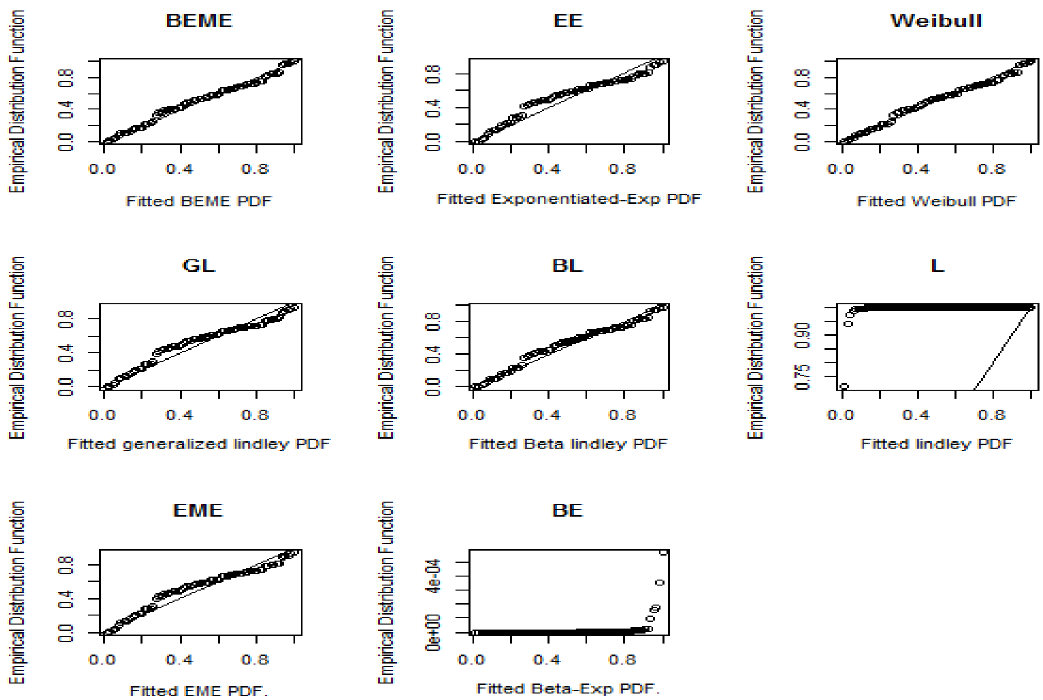

Fig. 5. p-p graphs for first data set

Second Data Set The following set of data, introduced by Kundu and Raqab [12], presents the moderately skewed to left trend of the gauge lengths of 20 mm, and the observations are: 1.312, 1.314, 1.479, 1.552, 1.700, 1.803, 1.861, 1.865, 1.944, 1.958, 1.966, 1.997, 2.006, 2.021, 2.027, 2.055, 2.063, 2.098, 2.140, 2.179, 2.224, 2.240, 2.253, 2.270, 2.272, 2.274, 2.301, 2.301, 2.359, 2.382, 2.382, 2.426, 2.434, 2.435, 2.478, 2.490, 2.511, 2.514, 2.535, 2.554, 2.566, 2.570, 2.586, 2.629, 2.633, 2.642, 2.648, 2.684, 2.697, 2.726, 2.770, 2.773, 2.800, 2.809, 2.818, 2.821, 2.848, 2.880, 2.809, 2.818, 2.821, 2.848, 2.880, 2.954, 3.012, 3.067, 3.084, 3.090, 3.096, 3.128, 3.233, 3.433, 3.585, 3.585.

Table 3. Parameters estimates, log-likelihood

Model	α **(S.E)**	**β (S.E)**	**a (S.E)**	**B (S.E)**	**-ln(L)**	**k-s value**
BEME	1.56	0.1734	3.175	216.67		
(α,β,a,b)	(1.72)	(0.089)	(6.56)	(7.37)	101.56	0.0595
GL	64.83	2.29				
	(23.45)	(0.177)	1	1	116.48	0.092
BL	0.184		13.45	95.14	104.88	0.064
	(0.074)	--	(2.58)	(61.03)		
	1.36		84.97	1.81		
BE	(1.01)	--	(138.13)	(2.22)	112.62	0.77
Weibull	5.74	0.371				
	(0.507)	(0.0079)	1	1	103.06	0.084
L	0.648					
	(0.087)	1	1	1	256.38	0.388
EE	2.019	89.44				
	(0.132)	(0.690)	1	1	117.6	0.095
EME	32.31	2.32				
(α,β,1,1)	(10.73)	(0.183)	1	1	115.9	0.094

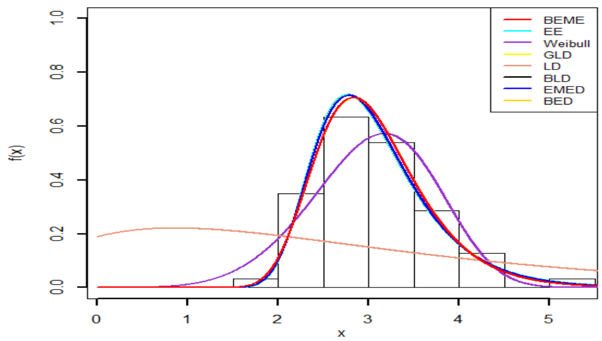

Fig. 6. PDF graphs for Second data set

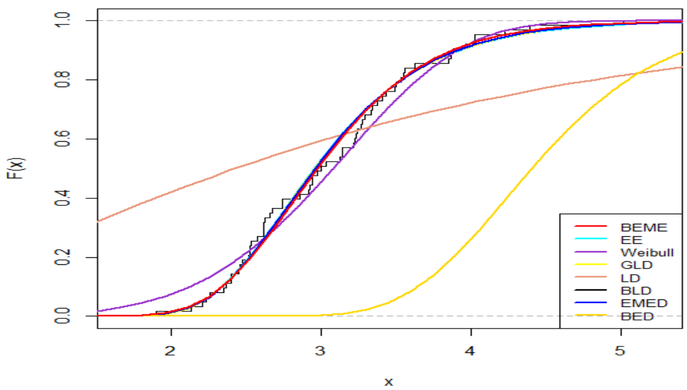

Fig. 7. CDF graphs for Second data set

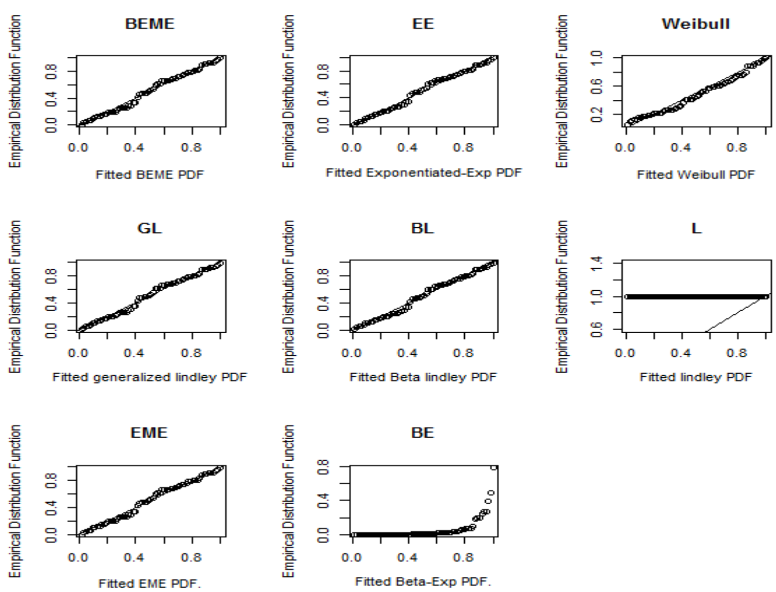

Fig. 8. p-p graphs for Second data set

Third Data Set: The following extreme skewed to right data, discussed by Murthy et al. [13], presents the failure times of 50 components and the observations are: 0.102, 0.061, 0.074, 0.192, 0.254, 0.262, 0.379, 0.590, 1.228, 1.600,0.381, 0.538, 0.570, 0.574, 8.022, 9.337, 0.618, 0.645, 0.961, 10.940, 11.020, , 3.147, 3.625, 3.704, 3.931, 4.073, 4.393, 4.534, 4.893, 6.274, 6.816,13.880, 2.006, 0.078, 0.086, 0.148, 0.183,7.896, 7.904, 14.730, 15.080, 2.054, 2.804, 0.103, 0.114, 0.116, 0.036, 0.058, 3.058, 3.076 [14-15].

Table 4. Parameters estimates, log-likelihood

n = 50 **Model**	α **(S.E)**	**β** **(S.E)**	**a** **(S.E)**	**B** **(S.E)**	**-2 ln(L)**	**k-s value**
BEME	4.61	2.25	0.088	0.0959	196.1	
(α,β,a,b)	(0.016)	(0.013)	(0.015)	(0.017)		0.0862
GL	0.411	0.296			207.96	
	(0.071)	(0.054)	1	1		0.163
BL	0.424		0.426	0.281	207.3	
	(0.076)	--	(0.364)	(0.53)		0.163
BE	0.523		0.717	1.81	204.7	
	(1.01)	--	(138.13)	(2.22)		0.37
Weibull	0.661	0.395			204.72	
	(0.074)	(0.089)	1	1		0.127
L	0.449				241.56	
	(0.051)	1	1	1		0.342
EE	0.194	0.537			204.74	
	(0.043)	(0.090)	1	1		0.142
EME	0.263	0.259			204.54	
(α,β,1,1)	(0.0435)	(0.055)	1	1		0.094

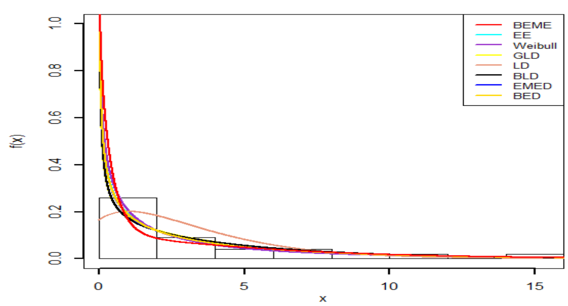

Fig. 9. PDF graphs for third data set

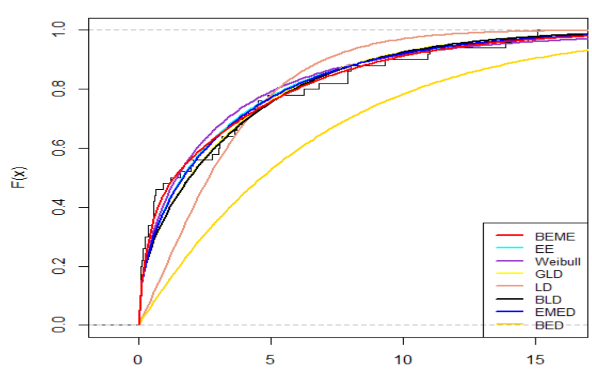

Fig. 10. CDF graphs for third data set

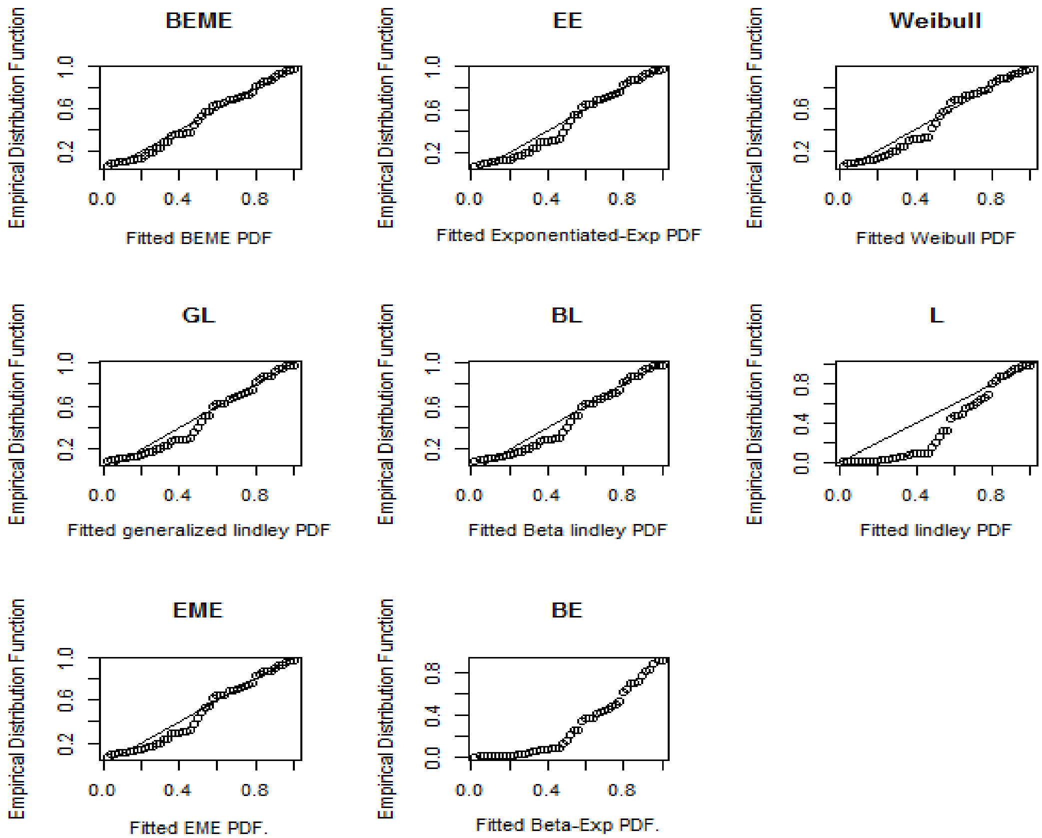

Fig. 11. P-P graphs for third data set

9 Conclusion

A new generalization of EME distribution is derived and studied. We obtain the structural properties including moments, quantile expression, reliability measures such as hrf and the other reverse hrf, the Bonferroni and the Lorenz curves and mean deviations. We have also conducted some numerical study. We have also found the important measures of entropy such as the measure of Renyi's entropy and the measure of s-entropy. We check the flexibility of model by applying this new model to three real data sets of different fields.

Competing Interests

Authors have declared that no competing interests exist.

References

[1] Eugene N, Lee C, Famoye F. Beta-normal distribution and its applications. Communications in Statistics - Theory and Methods. 2002;31(4):497-512.

[2] Jones MC. Families of distributions arising from distributions of order statistics. Test. 2004;13:1–43.

[3] Nadarajah S, Kotz S. The beta Gumbel distribution. Mathematical Problems in Engineering. 2004;10: 323-332.

[4] Nadarajah S, Gupta AK. The beta Fréchet distribution. Far East Journal of Theorical Statistics. 2004;15: 15-24.

[5] Pescim RR, Demétrio CG, Cordeiro GM, Ortega EM, Urbano MR. The beta generalized half-normal distribution. Computational Statistics & Data Analysis. 2010;54(4):945-957.

[6] Barreto-Souza W, Cordeiro GM, Simas AB. Some results for beta Fréchet distribution. Communications in Statistics—Theory and Methods. 2011;40(5):798-811.

[7] Dara ST. Recent Advances in Moment Distributions and their Hazard Rate. Ph.D. Thesis. National College of Business Administration and Economics, Lahore, Pakistan; 2012.

[8] Hasnain SA, Iqbal Z, Ahmad M. On exponentiated moment exponential distribution. Pakistan Journal of Statistics; 2015.

[9] Rényi A. On measures of information and entropy. In Proceedings of the 4th Berkeley symposium on mathematics, statistics and probability. 1961;1(547).

[10] Shannon CE, Weaver W. The mathematical theory of communication. Univ. Illinois press, Urbana. 1949;I, 11, 117.

[11] Nicholas MD, Padgett WJ. A bootstrap control chart for Weibull percentiles, Quality and Reliability Engineering International. 2006;22:141-151.

[12] Kundu D, Raqab MZ. Estimation of R= P (Y< X) for three-parameter Weibull distribution. Statistics & Probability Letters. 2009;79(17):1839-1846.

[13] Murthy DNP, Xie M, Jiang R. Weibull Models, John Wiley & Sons Inc., Hoboken, NJ; 2004.

[14] Nadarajah S. Explicit expressions for moments of order statistics. Statistics and Probability Letters. 2008;78:196-205.

[15] Nadarajah S, Kotz S. The beta exponential distribution. Reliability Engineering and System Safety. 2005;91:689-697.

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