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Hadamard and Fejér Hadamard Inequalities for $(h-m)-$Convex Functions Via Fractional Integral Containing the Generalized Mittag-Leffler Function

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Authors' contributions

This work was carried out in collaboration between all authors. All authors read and approved the final manuscript.

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ABSTRACT

The Hadamard and the Fejér Hadamard integral inequalities for $(h-m)-$convex functions via generalized fractional integral operators involving the generalized Mittag-Leffler function are established. In particular several known results are mentioned.

Keywords: RL fractional integrals; generalized Mittag-Leffler function; generalized fractional integrals; $(h-m)-$convex functions.

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1 INTRODUCTION AND PRELIMINARIES

The convexity of functions is a basic concept in mathematics, its extensions and generalizations have been considered in various directions using novel and innovative techniques. For example the $(h-m)-$convexity is the generalization of convexity that contains $h-$convexity, $(\alpha, m)-$convexity, $m-$convexity, $s-$convexity defined on the right half of real line including zero (see, [1, 2] and references there in).

Definition 1.1. Let $J \subseteq \mathbb{R}$ be an interval containing $(0,1)$ and let $h : J \to \mathbb{R}$ be a non-negative function. We say that $f : [0,b] \to \mathbb{R}$ is a $(h-m)-$convex function, if f is non-negative and for all $x, y \in [0,b], m \in [0,1]$ and $\alpha \in (0,1)$, one has

$$f(\alpha x + m(1-\alpha)y) \leq h(\alpha)f(x) + mh(1-\alpha)f(y).$$

For suitable choice of h and m, class of $(h-m)-$convex functions is reduces to the different known classes of convex functions defined on $[0,b]$.

Fractional Calculus is a field of mathematical study that grows out of the traditional definitions of the calculus integral and derivative operators in the same way fractional exponents is an outgrowth of exponents with integer value. Most of the mathematical theory applicable to the study of fractional calculus was developed prior to the turn of the 20th century.

The Mittag-Leffler function is an important function that finds widespread use in the world of fractional calculus. Just as the exponential naturally arises out of the solution to integer order differential equations, the Mittag-Leffler function plays an analogous role in the solution of non-integer order differential equations. In fact, the exponential function itself is a very specific form, one of an infinite set, of this seemingly ubiquitous function (see,[3],[4]).

Definition 1.2. [5] Let $\alpha, \beta, k, l, \gamma$ be positive real numbers and $\omega \in \mathbb{R}$. Then the generalized fractional integral operator containing Mittag-Leffler function for a real valued continuous function f is defined by

$$\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+} f\right)(x) := \int_a^x (x-t)^{\beta-1} E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega(x-t)^{\alpha}) f(t)dt$$

and

$$\left(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,b^-} f\right)(x) := \int_x^b (t-x)^{\beta-1} E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega(t-x)^{\alpha}) f(t)dt.$$

It is also common to represent the Mittag-Leffler function as

$$E^{\gamma,\delta,k}_{\alpha,\beta,l}(t) = \sum_{n=0}^{\infty} \frac{(\gamma)_{kn} t^n}{\Gamma(\alpha n + \beta)(\delta)_{tn}}, \qquad (1.1)$$

where $(a)_n = a(a+1)...(a+n-1), (a)_0 = 1$. This is the more generalized form of the Mittag-Leffler function. If $\delta = l = 1$, then integral operator $\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+}$ reduces to an integral operator $\epsilon^{\gamma,1,k}_{\alpha,\beta,1,\omega,a^+}$containing generalized Mittag-Leffler function $E^{\gamma,1,k}_{\alpha,\beta,1}$ introduced by Srivastava and Tomovski [6]. Along with $\delta = l = 1$ in addition if $k = 1$ then reduces to an integral operator defined by Prabhakar [7] containing Mittag-Leffler function $E^{\gamma}_{\alpha,\beta}$. For $\omega = 0$, integral operator $\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega,a^+}$ reduces to the Riemann-Liouville (RL) fractional integral operator.

Let $f \in L_1[a,b]$. Then left and right RL fractional integrals $I^{\alpha}_{a^+} f(x)$ and $I^{\alpha}_{b^-} f(x)$ of order $\alpha \in \mathbb{R}$ $(\alpha > 0)$ are defined by

$$I^{\alpha}_{a^+} f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)dt}{(x-t)^{1-\alpha}}, \quad x > a,$$

and

$$I^{\alpha}_{b^-} f(x) := \frac{1}{\Gamma(\alpha)} \int_x^b \frac{f(t)dt}{(t-x)^{1-\alpha}}, \quad x < b$$

respectively. Here $\Gamma(\alpha)$ is the Euler's Gamma function and $I^0_{a^+} f(x) = I^0_{b^-} f(x) = f(x)$.

Fractional integral inequalities are useful in establishing the uniqueness of solutions of fractional differential equations. A lot of work in fractional calculus in these days reflects its importance in almost all fields of mathematics, physics, information technology and other sciences [8, 9, 10, 11, 7, 12, 11, 13, 14, 15, 16, 17].

In this paper we have to prove the Hadamard and the Fejér-Hadamard type integral inequalities for $(h-m)-$convex functions via generalized fractional integral operator containing Mittag-Leffler function. Many known results have been produced from this inequality. An analogue Hadamard inequality is established as well as related results are mentioned. A version of Fejér Hadamard inequality is also established to summarize the results.

2 MAIN RESULTS

First we give the generalized fractional integral Hadamarad inequality for $(h-m)-$convex functions.

Theorem 2.1. *Let $f:[0,\infty)\to\mathbb{R}$ be an integrable and $(h-m)-$convex function with $m\in(0,1]$. Then the following inequality for generalized fractional integral holds*

$$\begin{aligned}
& f\left(\frac{bm+a}{2}\right)(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega^o,a^+}1)(mb) \qquad (2.1)\\
& \leq h\left(\frac{1}{2}\right)\left[m^{\beta+1}(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega^o,b^-}f)\left(\frac{a}{m}\right)+(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega^o,a^+}f)(mb)\right]\\
& \leq h\left(\frac{1}{2}\right)\left\{\left[m^2 f\left(\frac{a}{m^2}\right)+mf(b)\right]\int_0^1 t^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega t^\alpha)h(1-t)dt\right.\\
& \left.+[mf(b)+f(a)]\int_0^1 t^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega t^\alpha)h(t)dt\right\},
\end{aligned}$$

where $\omega^o=\frac{\omega}{(bm-a)^\alpha}$.

Proof. Using that f is $(h-m)-$convex we can have

$$f\left(\frac{xm+y}{2}\right)\leq h\left(\frac{1}{2}\right)(mf(x)+f(y)). \qquad (2.2)$$

Setting $x=(1-t)\frac{a}{m}+tb$ and $y=m(1-t)b+ta$, then integrating over $[0,1]$ after multiply with $t^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega t^\alpha)$ we get

$$\begin{aligned}
& f\left(\frac{bm+a}{2}\right)\int_0^1 t^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega t^\alpha)dt\\
& \leq h\left(\frac{1}{2}\right)\left[\int_0^1 t^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega t^\alpha)mf\left((1-t)\frac{a}{m}+tb\right)dt\right.\\
& \left.+\int_0^1 t^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega t^\alpha)f(m(1-t)b+ta)dt\right].
\end{aligned}$$

By substituting $w=(1-t)\frac{a}{m}+tb$ and $z=m(1-t)b+ta$ one can have

$$\begin{aligned}
& f\left(\frac{bm+a}{2}\right)(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega^o,a^+}1)(mb) \qquad (2.3)\\
& \leq h\left(\frac{1}{2}\right)\left[m^{\beta+1}(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega^o,b^-}f)\left(\frac{a}{m}\right)+(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega^o,a^+}f)(mb)\right].
\end{aligned}$$

This completes the proof of first inequality in (2.1). For the second inequality $(h-m)-$convexity of f also gives

$$mf\left((1-t)\frac{a}{m}+tb\right)+f(m(1-t)b+ta)$$
$$\leq m^2h(1-t)f\left(\frac{a}{m^2}\right)+mh(t)f(b)+mh(1-t)f(b)+h(t)f(a).$$

Multiplying both sides of above inequality with $h\left(\frac{1}{2}\right)t^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega t^{\alpha})$ and integrating over $[0,1]$, we have

$$h\left(\frac{1}{2}\right)\left[m^{\beta+1}(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega^o,b^-}f)\left(\frac{a}{m}\right)+(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega^o,a^+}f)(mb)\right].$$
$$\leq h\left(\frac{1}{2}\right)\left\{\left[m^2f\left(\frac{a}{m^2}\right)+mf(b)\right]\int_0^1 t^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega t^{\alpha})h(1-t)dt\right.$$
$$\left.+[mf(b)+f(a)]\int_0^1 t^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega t^{\alpha})h(t)dt\right\}.$$

Combining it with (2.3) we get (2.1) which was required to prove. □

Several known results are special cases of the above generalized fractional Hadamard inequality comprise in the following remark.

Remark 2.1.

i) If we take $h(t)=t$ and $m=1$ in above theorem, then we get [18, Theorem 2.1].
ii) If we take $h(t)=t$ in above theorem, then we get [10, Theorem 3].
iii) If we take $h(t)=t$ and $\omega=0$ in above theorem, then we get [11, Theorem 2.1].
iv) If we take $h(t)=t$, $m=1$ and $\omega=0$ in above theorem, then we get [19, Theorem 2].
v) If we take $h(t)=t$, $m=1$, $\beta=1$ and $\omega=0$ in above theorem, then we get the Hadamard inequality.

In the following we prove another analogue version of the Hadamard inequality for generalized fractional integrals.

Theorem 2.2. *Let $f:[0,\infty)\to\mathbb{R}$ be an integrable and $(h-m)-$convex function with $m\in(0,1]$. Then the following inequality for generalized fractional integral holds*

$$f\left(\frac{a+bm}{2}\right)(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega^o,a^+}1)(mb) \tag{2.4}$$
$$\leq h\left(\frac{1}{2}\right)\left[(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega^o,(\frac{a+bm}{2})^+}f)(mb)+m^{\beta+1}(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega^o,(\frac{a+bm}{2m})^-}f)\left(\frac{a}{m}\right)\right]$$
$$\leq h\left(\frac{1}{2}\right)\left\{\left(m^2f\left(\frac{a}{m^2}\right)+mf(b)\right)\int_0^1 t^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega t^{\alpha})h\left(\frac{2-t}{2}\right)dt\right.$$
$$\left.+[mf(b)+f(a)]\int_0^1 t^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega t^{\alpha})h\left(\frac{t}{2}\right)dt\right\}.$$

Proof. By putting $y=\frac{t}{2}b+\frac{(2-t)}{2}\frac{a}{m}$ and $x=\frac{t}{2}a+m\frac{(2-t)}{2}b$ in (2.2) where $t\in[0,1]$, we have

$$f\left(\frac{a+bm}{2}\right)\leq h\left(\frac{1}{2}\right)\left[f\left(\frac{t}{2}a+m\frac{(2-t)}{2}b\right)+mf\left(\frac{t}{2}b+\frac{(2-t)}{2}\frac{a}{m}\right)\right].$$

Multiplying both sides of above inequality with $t^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega t^{\alpha})$ and integrating over $[0,1]$, we have

$$f\left(\frac{a+bm}{2}\right)\int_0^1 t^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega t^{\alpha})dt$$
$$\leq h\left(\frac{1}{2}\right)\left[\int_0^1 t^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega t^{\alpha})mf\left(\frac{t}{2}a+m\frac{(2-t)}{2}b\right)dt\right.$$
$$\left.+\int_0^1 t^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega t^{\alpha})f\left(\frac{t}{2}b+\frac{(2-t)}{2}\frac{a}{m}\right)dt\right].$$

By change of variables one can have

$$f\left(\frac{a+bm}{2}\right)(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega^o,a^+}1)(mb) \tag{2.5}$$
$$\leq h\left(\frac{1}{2}\right)\left[(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega^o,(\frac{a+bm}{2})^+}f)(mb)+m^{\beta+1}(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega^o,(\frac{a+bm}{2m})^-}f)\left(\frac{a}{m}\right)\right].$$

Now by using the $(h-m)-$convexity of f one can get

$$f\left(\frac{t}{2}a+m\frac{(2-t)}{2}b\right)+mf\left(\frac{t}{2}b+\frac{(2-t)}{2}\frac{a}{m}\right).$$
$$\leq h\left(\frac{t}{2}\right)f(a)+mh\left(\frac{2-t}{2}\right)f(b)+mh\left(\frac{t}{2}\right)f(b)+m^2h\left(\frac{2-t}{2}\right)f\left(\frac{a}{m^2}\right).$$

Multiplying both sides of above inequality with $h\left(\frac{1}{2}\right)t^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega t^{\alpha})$ and integrating over $[0,1]$, we have

$$h\left(\frac{1}{2}\right)\left[(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega^o,(\frac{a+bm}{2})^+}f)(mb)+m^{\beta+1}(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega^o,(\frac{a+bm}{2m})^-}f)\left(\frac{a}{m}\right)\right]$$
$$\leq h\left(\frac{1}{2}\right)\left\{\left(m^2f\left(\frac{a}{m^2}\right)+mf(b)\right)\int_0^1 t^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega t^{\alpha})h\left(\frac{2-t}{2}\right)dt\right.$$
$$\left.+[mf(b)+f(a)]\int_0^1 t^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega t^{\alpha})h\left(\frac{t}{2}\right)dt\right\}.$$

Combining it with (2.5) we get (2.4) which was the required inequality. □

Corollary 2.3. *If we take $h(t)=t$ and $m=1$ in above theorem, then we get the following inequality analogue to the Hadamard inequality [18, Theorem 2.1] for convex functions via generalized fractional integrals*

$$f\left(\frac{b+a}{2}\right)(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega^o,a^+}1)(b)\leq\frac{1}{2}\left[(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega^o,(\frac{a+b}{2})^-}f)(a)+(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega^o,(\frac{a+b}{2})^+}f)(b)\right]$$
$$\frac{1}{2}[f(a)+f(b)](\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega^o,b^-}1)(a).$$

If we take $h(t)=t$, $m=1$ and $\omega=0$ in above theorem, then we get the following result for Riemann-Liouville fractional integral.

Corollary 2.4. *[20] Let $f:[a,b]\to\mathbb{R}$ be a function with $0\leq a<b$ and $f\in L_1[a,b]$. If f is a convex function on $[a,b]$, then the following inequalities for fractional integrals hold*

$$f\left(\frac{b+a}{2}\right)\leq\frac{2^{\alpha+1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}[I^{\alpha}_{(\frac{a+b}{2})^+}f(b)+I^{\alpha}_{(\frac{a+b}{2})^-}f(a)]\leq\frac{1}{2}[f(a)+f(b)].$$

In the following we give Fejér Hadamard inequality for $(h,m)-$convex functions via generalized fractional integral operator.

Theorem 2.5. *Let $f:[a,b]\to\mathbb{R}$ be a $(h-m)-$convex function with $0\le a<b$ and $f\in L_1[a,b]$. Also, let $g:[a,b]\to\mathbb{R}$ be a function which is non-negative, integrable and symmetric about $\frac{a+mb}{2}$. If $f(mb+a-mx)=f(x)$, then the following inequalities for generalized fractional integrals hold*

$$f\left(\frac{bm+a}{2}\right)(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega^o,b^-}g)\left(\frac{a}{m}\right)\le h\left(\frac{1}{2}\right)(m+1)(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega^o,b^-}fg)\left(\frac{a}{m}\right) \tag{2.6}$$
$$\le h\left(\frac{1}{2}\right)\left\{\left[m^2f\left(\frac{a}{m^2}\right)+mf(b)\right]\int_0^1 t^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega t^\alpha)h(1-t)dt\right.$$
$$\left.+[mf(b)+f(a)]\int_0^1 t^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega t^\alpha)h(t)dt\right\},$$

where $\omega^o=\frac{\omega}{(bm-a)^\alpha}$.

Proof. As f is $(h-m)-$convex, therefore for $t=\frac{1}{2}$; $(1-t)\frac{a}{m}+tb, m(1-t)b+ta\in[a,b]$ and we have

$$f\left(\frac{bm+a}{2}\right)\le h\left(\frac{1}{2}\right)\left[mf\left((1-t)\frac{a}{m}+tb\right)+f(m(1-t)b+ta)\right].$$

Multiplying both sides of above inequality with $t^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega t^\alpha)g\left(tb+(1-t)\frac{a}{m}\right)$ and integrating over $[0,1]$, we have

$$f\left(\frac{bm+a}{2}\right)\int_0^1 t^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega t^\alpha)g\left(tb+(1-t)\frac{a}{m}\right)dt$$
$$\le h\left(\frac{1}{2}\right)\left[\int_0^1 t^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega t^\alpha)g\left(tb+(1-t)\frac{a}{m}\right)mf\left((1-t)\frac{a}{m}+tb\right)dt\right.$$
$$\left.+\int_0^1 t^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega t^\alpha)g\left(tb+(1-t)\frac{a}{m}\right)f(m(1-t)b+ta)dt\right].$$

If we set $x=(1-t)\frac{a}{m}+tb$ and use the given condition $f(mb+a-mx)=f(x)$ one can have

$$f\left(\frac{bm+a}{2}\right)(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega^o,b^-}g)\left(\frac{a}{m}\right)\le h\left(\frac{1}{2}\right)(m+1)(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega^o,b^-}fg)\left(\frac{a}{m}\right). \tag{2.7}$$

This completes the proof of first inequality in (2.6). For the second inequality using $(h-m)-$convexity of f we have

$$mf\left((1-t)\frac{a}{m}+tb\right)+f(m(1-t)b+ta)$$
$$\le m^2h(1-t)f\left(\frac{a}{m^2}\right)+mh(t)f(b)+mh(1-t)f(b)+h(t)f(a).$$

Multiplying both sides of above inequality with $h\left(\frac{1}{2}\right)t^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega t^\alpha)g\left(tb+(1-t)\frac{a}{m}\right)$ and integrating over $[0,1]$, we have

$$h\left(\frac{1}{2}\right)(m+1)(\epsilon^{\gamma,\delta,k}_{\alpha,\beta,l,\omega^o,b^-}fg)\left(\frac{a}{m}\right)$$
$$\le h\left(\frac{1}{2}\right)\left\{\left[m^2f\left(\frac{a}{m^2}\right)+mf(b)\right]\int_0^1 t^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega t^\alpha)h(1-t)g\left(tb+(1-t)\frac{a}{m}\right)dt\right.$$
$$\left.+[mf(b)+f(a)]\int_0^1 t^{\beta-1}E^{\gamma,\delta,k}_{\alpha,\beta,l}(\omega t^\alpha)h(t)g\left(tb+(1-t)\frac{a}{m}\right)dt\right\}.$$

Combining it with (2.7) we get (2.6) which was the required inequality. □

3 CONCLUDING REMARKS

Established results are the generalized fractional integral inequalities may be the useful restrictions onto the uniqueness of solutions of some generalized fractional boundary value problems. Actually the Mittag-Lefller function appears in (1.1) corresponds to different novel generalizations of classical Riemann-Liouville fractional integrals. Therefore obtained results hold for all these fractional integral operators. Also $(h, m)-$convexity in special cases give a cluster of the fractional integral inequalities for $h-$convex, $(s - m)$convex, $m-$convex, $m-$convex, convex functions. The Fejér Hadamard inequality summarizes all the discussed results in a very nice compact form. We hope this work will attract the readers of this journal comprehensively to the field of convex analysis via fractional calculus.

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COMPETING INTERESTS

Authors have declared that no competing interests exist.

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