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On the Buckling Modes and Buckling Load of an Infinitely Long but Harmonically Imperfect Column Lying on Cubic – Quintic Foundation

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Authors' contributions

The problem was proposed by author AME who is also the head of the research team. All the authors, namely AME, JUC, WIO and GEO, solved the problem individually. Later, all the authors met to discuss the various solutions and took a decision on the final draft of the solution. The final draft was typed by author JUC. She also did the numerical computation and graphical plots. All authors read and approved the final manuscript.

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Abstract

This paper utilizes perturbation and asymptotic techniques to discuss and obtain, analytically, the buckling modes and buckling load of a harmonically imperfect column lying on an elastic foundation that has cubic – quintic nonlinearity. Two slightly different approaches are here utilized. In the first approach, the perturbation parameter is a component of the displacement while in the second approach, the perturbation is a component of the load. In the final assessment, results from both approaches are seen to be in good agreement. The results are however observed to be implicit in the load parameter and are valid asymptotically as long as these perturbation parameters are small relative to unity.

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1 Introduction

In this paper, a perturbation scheme in asymptotic series expansions is developed in determining the static buckling load and buckling modes of an infinitely long but harmonically imperfect column lying on a cubic – quintic nonlinear elastic foundation where the column is trapped by a static load of magnitude P. It is to be recalled that, as far as investigations concerning columns are concerned, majority of the existing research findings have tended to favour columns lying on nonlinear cubic elastic foundations [1, 2, 3] to the exclusion of most other nonlinear elastic foundations. In this study, we intend to stretch the analysis to the case where the foundation has a cubic – quintic nonlinearity.

We remark that investigations on columns lying on cubic – quintic non – linear elastic foundations are rare, for reasons, perhaps, attributed to the inherent excessive nonlinearity which makes any analytical solution of the problem difficult and cumbersome. However, it is to be recalled that Elishakoff [4] had earlier studied the buckling of similar columns lying on cubic foundations as well as those lying on quadratic – cubic non – linear elastic foundations. It is our intention in this analysis to extend Elishakoff's analysis to the case of a column lying on cubic – quintic non – linear elastic foundations.

Generally, investigations on buckling, both static and dynamic, have tended to attract and occupy a prominent attention amongst the research community for a long time now. In this respect, mention is here made of an investigation by Reda and Forbes [5], Priyadarsini et al. [6], Chitra and Priyadarsini [7], Mcshane et al. [8], Kolakowski [9, 10] and Patil et al. [11], among others.

2 Governing Equation

The normal displacement $W(X)$ of the column, subjected to the applied load P, satisfies the non – homogeneous equation

$$EI\frac{d^4W}{dX^4} + 2P\frac{d^2W}{dX^2} + k_1W + \alpha k_2W^3 - \beta_1k_3W^5 = -2P\frac{d^2\overline{W}}{dX^2}, \quad -\infty < X < \infty \tag{2.1}$$

where $\left|w(x)\right| < \infty$ as $x \rightarrow \pm\infty$.

In general, we demand that the displacement w(x) be bounded for all values of x.

Here X is the spatial coordinate, EI is the bending stiffness, where E and I are Young's modulus and moment of inertia respectively and $\overline{W}$ is the twice differentiable stress – free harmonic imperfection. The cubic – quintic nonlinear elastic foundation exerts a force per unit length given by $k_1W + \alpha k_2W^3 - \beta_1k_3W^5$ on the column, while α and β_1 are the imperfection – sensitivity factors which are to be carefully chosen so that the column becomes imperfection – sensitive and k_1, k_2 and k_3 are positive constants. In this formulation, we have neglected all nonlinearities greater than quintic while all nonlinear derivatives are neglected.

In order to nondimensionalize the equation, the following nondimensional quantities are now assumed.

$$X = \left(\frac{EI}{k_1}\right)^{\frac{1}{4}}x, \quad W = \left(\frac{k_1}{k_2}\right)^{\frac{1}{2}}w, \quad \overline{W} = \epsilon\left(\frac{k_1}{k_2}\right)^{\frac{1}{2}}\overline{w}, \quad \beta = \left(\frac{\beta_1k_1k_3}{k_2^2}\right), \quad P = 2\lambda(EIk_1)^{\frac{1}{2}}$$

Here, ϵ and λ satisfy the inequalities $0 < \epsilon \ll 1, 0 < \lambda < 1$, and the nondimensional form of the equation is

$$\frac{d^4w}{dx^4} + 2\lambda\frac{d^2w}{dx^2} + w + \alpha w^3 - \beta w^5 = -2\lambda\epsilon\frac{d^2\overline{w}}{dx^2}, \quad -\infty < x < \infty \tag{2.2}$$

In general, the amplitude of imperfection ϵ satisfies the inequality $|\epsilon| \ll 1$, but in this analysis we shall let ϵ satisfy $0 < \epsilon \ll 1$.

We shall solve the equation in two slightly different approaches whereby, in the first approach, we adopt the perturbation and asymptotic parameter as a component of displacement whereas in the second approach, we adopt the perturbation parameter as a component of the applied load. In this latter case, we shall let $\lambda = 1 - \frac{\bar{\varepsilon}^2}{2}$, for $0 < \bar{\varepsilon} \ll 1$, where λ is the nondimensional load amplitude. In both cases, we aim at first determining a uniformly valid asymptotic expression of the normal displacement subsequent upon which the static buckling load λ_S, is next determined. The static buckling load λ_S, as in [1 – 3], is defined as the maximum value of the load amplitude λ that emanates from the origin of the load – displacement graphical configuration of the loading system. It is to be recalled that perturbation techniques in the load and displacement parameters used in this work were judiciously utilized by Amazigo [2] when he investigated the buckling of an infinitely long column with harmonic imperfection lying on a non – linear cubic elastic foundation. His method is hereby extended to study the case of a column lying on a cubic – quintic non – linear elastic foundation.

3 Solution of (2.2) Using Displacement as Perturbation Parameter

Since the imperfection is harmonic, we let

$$\bar{w} = \cos nx, \qquad n = 1, 2, 3, \dots \tag{3.1}$$

Assuming that the displacement must be in the shape of imperfection, we let

$$w(x) = \cos nx, \tag{3.2}$$

The equation satisfied by the perfect linear structure is

$$\frac{d^4 w}{dx^4} + 2\lambda \frac{d^2 w}{dx^2} + w = 0 \tag{3.3}$$

The resultant equation when (3.2) is substituted in (3.3) is

$$(n^4 - 2n^2\lambda + 1) = 0, \quad \lambda = \frac{1}{2n^2}(n^4 + 1) \tag{3.4}$$

The least value of λ in (3.4) is obtained when $n = 1$ and for this the classical buckling load λ_C is

$$\lambda_C = 1 \tag{3.5}$$

For the solution of (2.2), it is necessary to let

$$w(x) = \bar{\varepsilon}\cos x + v(x) \tag{3.6}$$

It is here assumed that the average value of $v(x)\cos x$ vanishes over the interval of definition of x, that is

$$< v(x) cosx > = 0 \tag{3.7}$$

where, $< \cdots >$ denotes the average of $v(x)cosx$.Thus, with w known, $\bar{\varepsilon}$ is uniquely defined.

Let

$$v(x) = \sum_{m=2}^{\infty} \bar{\varepsilon}^m v_m, \qquad \lambda\epsilon = \sum_{m=1}^{\infty} \bar{\varepsilon}^m Q_m \tag{3.8}$$

In order to solve (2.2), using (3.2), equations (3.8) are now substituted into (2.2) and thereafter, we equate the coefficients of powers of $\bar{\varepsilon}$ to get

$$\boldsymbol{O}(\bar{\varepsilon}):\quad 2(1-\lambda)cosx = 2Q_1\cos x \tag{3.9}$$

$$\boldsymbol{O}(\bar{\varepsilon}^2):\quad Mv_2 \equiv \frac{d^4v_2}{dx^4} + 2\lambda\frac{d^2v_2}{dx^2} + v_2 = 2Q_2\cos x \tag{3.10}$$

$$\boldsymbol{O}(\bar{\varepsilon}^3):\quad Mv_3 = 2Q_3\cos x - \alpha\cos^3 x \tag{3.11}$$

$$\boldsymbol{O}(\bar{\varepsilon}^4):\quad Mv_4 = 2Q_4\cos x - 3\alpha v_2\cos^2 x \tag{3.12}$$

$$\boldsymbol{O}(\bar{\varepsilon}^5):\quad Mv_5 = 2Q_5\cos x - 3\alpha v_3\cos^2 x - 3\alpha v_2\cos x + \beta\cos^5 x \tag{3.13}$$

$$\boldsymbol{O}(\bar{\varepsilon}^6):\quad Mv_6 = 2Q_6\cos x - 3\alpha v_2\cos^4 x - 6\alpha v_2 v_3\cos x + 5\beta\, v_2\cos^4 x \tag{3.14}$$

$$\boldsymbol{O}(\bar{\varepsilon}^7):\quad Mv_7 = 2Q_7\cos x - \alpha[3v_5\cos^2 x + 6v_2v_4\cos x + 3v_3^2\cos x] + \beta v_3\cos^4 x \tag{3.15}$$

etc.

From (3.9), it is easily seen that

$$Q_1 = (1-\lambda),\qquad v_1 = 0 \tag{3.16}$$

On using the condition (3.7), it is seen that

$$Q_2 = 0,\qquad v_2 = 0 \tag{3.17}$$

On simplification, equation (3.11) becomes

$$Mv_3 = \left(2Q_3 - \frac{3\alpha}{4}\right)cosx - \frac{\alpha}{4}\cos 3x \tag{3.18}$$

On using the condition (3.7) on (3.18), it easily follows that

$$Q_3 = \frac{3\alpha}{8} \tag{3.19}$$

After solving the remaining equation in (3.18), we have

$$v_3 = \frac{-\alpha\cos 3x}{8(41-9\lambda)} \tag{3.20}$$

From (3.12), it easily follows that

$$Q_4 = v_4 = 0 \tag{3.21}$$

Equation (3.13) is next simplified to yield (using (3.17))

$$Mv_5 = \left(2Q_5 + \frac{5\beta}{8} + \frac{3\alpha^2}{32(41-9\lambda)}\right)\cos x + \left(\frac{15\beta}{16} + \frac{3\alpha^2}{16(41-9\lambda)}\right)\cos 3x + \left(\frac{\beta}{16} + \frac{3\alpha^2}{32(41-9\lambda)}\right)\cos 5x \tag{3.22}$$

On applying (3.7) in (3.22), this yields

$$Q_5 = -\frac{1}{2}\left(\frac{5\beta}{8} + \frac{3\alpha^2}{32(41-9\lambda)}\right) \tag{3.23}$$

The solution of the remaining equation in (3.22) is

$$v_5 = \frac{1}{32}\left(\beta + \frac{3\alpha^2}{(41-9\lambda)}\right)\left(\frac{\cos 3x}{(41-9\lambda)}\right) + \frac{1}{64}\left(2\beta + \frac{3\alpha^2}{(41-9\lambda)}\right)\left(\frac{\cos 5x}{(313-25\lambda)}\right) \tag{3.24}$$

After substituting in (3.14), we get

$$Q_6 = v_6 = 0 \tag{3.25}$$

Next, we substitute in (3.15) and simplify to get

$$\begin{aligned} Mv_7 = &\left[2Q_7 - \frac{3\alpha}{2}\left\{\frac{A_1}{2} + \frac{\alpha^2}{128(41-9\lambda)^2}\right\} + \frac{5\beta}{8}\right]\cos x + \left[\frac{3\beta}{8} - \frac{3\alpha}{2}\left(A_1 + \frac{A_2}{2}\right)\right]\cos 3x \\ &+ \left[\frac{\beta}{2} - \frac{3\alpha}{2}\left\{\left(A_2 + \frac{A_1}{2}\right) + \frac{\alpha^2}{256(41-9\lambda)^2}\right\}\right]\cos 5x \\ &+ \left[\frac{\beta}{8} - \frac{3\alpha}{2}\left\{\frac{A_2}{2} + \frac{\alpha^2}{256(41-9\lambda)^2}\right\}\right]\cos 7x \end{aligned} \tag{3.26a}$$

where,

$$A_1 = \frac{1}{32(41-9\lambda)}\left(\beta + \frac{3\alpha^2}{41-9\lambda}\right) \tag{3.26b}$$

$$A_2 = \frac{1}{64(313-25\lambda)}\left(2\beta + \frac{3\alpha^2}{41-9\lambda}\right) \tag{3.26c}$$

The condition (3.7) as applied to (3.26a) yields

$$Q_7 = \frac{1}{2}\left[\frac{3\alpha}{2}\left\{\frac{A_1}{2} + \frac{\alpha^2}{128(41-9\lambda)^2}\right\}\right] - \frac{5\beta}{8} \tag{3.26d}$$

The solution of the remaining equation in (3.26a) yields

$$\begin{aligned} v_7 = &\frac{1}{2}\left[\frac{3\beta}{8} - \frac{3\alpha}{2}\left(A_1 + \frac{A_2}{2}\right)\right]\left(\frac{\cos 3x}{41-9\lambda}\right) + \frac{1}{2}\left[\frac{\beta}{2} - \frac{3\alpha}{2}\left(A_2 + \frac{A_1}{2} + \frac{\alpha^2}{256(41-9\lambda)^2}\right)\right]\left(\frac{\cos 5x}{313-25\lambda}\right) \\ &+ \frac{1}{2}\left[\frac{\beta}{8} - \frac{3\alpha}{2}\left\{\frac{A_2}{2} + \frac{\alpha^2}{256(41-9\lambda)^2}\right\}\right]\left(\frac{\cos 7x}{1201-49\lambda}\right) \end{aligned} \tag{3.27}$$

Following (3.6), we can now write

$$\begin{aligned} w = &\bar{\varepsilon}\cos x - \frac{\alpha\bar{\varepsilon}^3\cos 3x}{8(41-9\lambda)} + \bar{\varepsilon}^5\left[\frac{1}{32}\left(\beta + \frac{3\alpha^2}{41-9\lambda}\right)\left(\frac{\cos 3x}{41-9\lambda}\right) + \frac{1}{64}\left(2\beta + \frac{3\alpha^2}{41-9\lambda}\right)\left(\frac{\cos 5x}{313-25\lambda}\right)\right] \\ &+ \frac{\bar{\varepsilon}^7}{2}\left[\left[\frac{3\beta}{8} - \frac{3\alpha}{2}\left(A_1 + \frac{A_2}{2}\right)\right]\left(\frac{\cos 3x}{41-9\lambda}\right)\right. \\ &+ \left[\frac{\beta}{2} - \frac{3\alpha}{2}\left(A_2 + \frac{A_1}{2} + \frac{\alpha^2}{256(41-9\lambda)^2}\right)\right]\left(\frac{\cos 5x}{313-25\lambda}\right) \\ &+ \left.\left[\frac{\beta}{8} - \frac{3\alpha}{2}\left\{\frac{A_2}{2} + \frac{\alpha^2}{256(41-9\lambda)^2}\right\}\right]\left(\frac{\cos 7x}{1201-49\lambda}\right)\right] + \cdots \end{aligned} \tag{3.28}$$

Similarly, we have (from (3.8))

$$\lambda\epsilon = \bar{\varepsilon}(1-\lambda) + \frac{3\alpha\bar{\varepsilon}^3}{8} - \frac{\bar{\varepsilon}^5}{2}\left(\frac{5\beta}{8} + \frac{3\alpha^2}{32(41-9\lambda)}\right) + \frac{\bar{\varepsilon}^7}{2}\left[\frac{3\alpha}{2}\left\{\frac{A_1}{2} + \frac{\alpha^2}{128(41-9\lambda)^2}\right\} - \frac{5\beta}{8}\right] + \cdots \tag{3.29}$$

To determine the static buckling load λ_S, we, as in [1 – 3], use the condition

$$\frac{d\lambda}{d\bar{\varepsilon}} = 0, \tag{3.30}$$

and get

$$(1-\lambda_S) + \frac{9\alpha\bar{\varepsilon}_S^2}{8} - \frac{5\bar{\varepsilon}_S^4}{2}\left(\frac{3\alpha^2}{32(41-9\lambda)} + \frac{5\beta}{8}\right) = 0 \tag{3.31}$$

On solving, this yields

$$\bar{\varepsilon}_S^2 = \frac{9\alpha}{40\left\{\frac{3\alpha^2}{32(41-9\lambda_S)} + \frac{5\beta}{8}\right\}}\left[1 - \sqrt{1 + \frac{512(1-\lambda_S)}{405\alpha^2\left\{\frac{3\alpha^2}{32(41-9\lambda_S)} + \frac{5\beta}{8}\right\}}}\right] \tag{3.32}$$

And

$$\therefore \quad \bar{\varepsilon}_S = \frac{3}{2\sqrt{10}}\sqrt{\frac{\alpha}{40\left\{\frac{3\alpha^2}{32(41-9\lambda_S)} + \frac{5\beta}{8}\right\}}}\left[1 - \left\{1 + \frac{512(1-\lambda_S)}{405\alpha^2\left\{\frac{3\alpha^2}{32(41-9\lambda_S)} + \frac{5\beta}{8}\right\}}\right\}^{\frac{1}{2}}\right]^{\frac{1}{2}} \tag{3.33}$$

The static buckling load λ_S is now obtained by evaluating (3.29) at $\lambda = \lambda_S$ and substituting for $\bar{\varepsilon}_S^2$ and $\bar{\varepsilon}_S$ from (3.32) and (3.33) respectively and this yields

$$\lambda_S\epsilon = \bar{\varepsilon}_S\left[(1-\lambda_S) + \bar{\varepsilon}_S^2\left\{\left\{\frac{3\alpha}{8} - \bar{\varepsilon}_S^2\left(\frac{3\alpha^2}{32(41-9\lambda_S)} + \frac{5\beta}{8}\right) + \frac{\bar{\varepsilon}_S^2}{2}\left\{\frac{3\alpha}{2}\left(\frac{A_1}{2} + \frac{\alpha^2}{128(41-9\lambda)^2}\right) - \frac{5\beta}{8}\right\}\right\}\right\}\right] \tag{3.34}$$

4 Solution of (2.2) with Load Component as Perturbation parameter

Here, we shall let

$$\lambda = 1 - \frac{\varepsilon^2}{2}, \quad 0 < \varepsilon < 1 \tag{4.1}$$

In this case, equation (2.2) becomes

$$\frac{d^4 w}{dx^4} + 2\frac{d^2 w}{dx^2} - \varepsilon^2\frac{d^2 w}{dx^2} + w + \alpha w^3 - \beta w^5 = -2\lambda\epsilon\frac{d^2\bar{w}}{dx^2} \tag{4.2}$$

Let

$$w(x) = \bar{b}\varepsilon\cos x + u(x), \qquad 0 < \bar{b} < 1 \tag{4.3}$$

Further let

$$u(x) = \sum_{m=2}^{\infty} \varepsilon^m u_m, \qquad \lambda\epsilon = \sum_{m=1}^{\infty} \varepsilon^m \gamma_m \tag{4.4}$$

Substituting for terms in (4.2) and equating the coefficients of powers of ε, yields

$$\boldsymbol{O}(\varepsilon):\ \ Nu_1 \equiv \frac{d^4u_2}{dx^4} + 2\lambda\frac{d^2u_2}{dx^2} + u_2 = 2\gamma_1 \cos x \tag{4.5}$$

$$\boldsymbol{O}(\varepsilon^2):\ \ Nu_2 = 2\gamma_2 \cos x \tag{4.6}$$

$$\boldsymbol{O}(\varepsilon^3):\ \ Nu_3 = -\bar{b}\cos x - \alpha\bar{b}^3\cos^3 x + 2\gamma_3 \cos x \tag{4.7}$$

$$\boldsymbol{O}(\varepsilon^4):\ \ Nu_4 = \frac{d^2u_2}{dx^2} - 3\bar{b}^2 u_2\, \alpha\cos^2 x + 2\gamma_4 \cos\ x \tag{4.8}$$

$$\boldsymbol{O}(\varepsilon^5):\ \ Nu_5 = \frac{d^2u_3}{dx^2} - 3\bar{b}^3 u_3\, \alpha\cos^2 x - 3\bar{b}\alpha u_2^2 \cos x + \beta\, \bar{b}^5\cos^5 x\ + 2\gamma_5 \cos\ x \tag{4.9}$$

$$\boldsymbol{O}(\varepsilon^6):\ \ Nu_6 = \frac{d^2u_4}{dx^2} - \alpha\{3\bar{b}^2 u_4 \cos^2 x + 6\bar{b}u_2u_3 \cos x - u_2^3\} - 6\alpha v_2 v_3 \cos x + 5\beta\, u_2\bar{b}^4\cos^4 x + 2\gamma_6 \cos\ x \tag{4.10}$$

$$\boldsymbol{O}(\varepsilon^7):\ \ Nu_7 = \frac{d^2u_5}{dx^2} - \alpha\{3\bar{b}^2 u_5 \cos^2 x + 3\bar{b}u_2 \cos x(u_3^2 + 2u_2u_4) + 3u_2^3 u_3\} + 5\beta\, u_3\bar{b}^4\cos^4 x + 2\gamma_7 \cos\ x \tag{4.11}$$

etc.

We shall still use the same orthogonality condition as (3.7). Thus, from (4.5), we get

$$\gamma_1 = 0, \qquad u_1 = 0 \tag{4.12a}$$

From (4.6), we get

$$\gamma_2 = 0, \qquad u_2 = 0 \tag{4.12b}$$

Equation (4.7) simplifies to

$$Nu_3 = \left(2\gamma_3 - \bar{b} - \frac{3\alpha\bar{b}^3}{4}\right) cosx - \frac{\alpha\bar{b}^3}{4}\, cos3x \tag{4.13}$$

Application of (3.7) in (4.13) yields

$$\gamma_3 = \frac{1}{2}\left(\bar{b} + \frac{\alpha\bar{b}^3}{4}\right) \tag{4.14a}$$

The solution of the remaining equation in (4.13) is

$$u_3 = \frac{\alpha\bar{b}^3}{32}\, cos3x \tag{4.14b}$$

Substituting for u_2 in (4.8) yields

$$\gamma_4 = 0, \qquad u_4 = 0 \tag{4.15}$$

Substituting for u_2 and u_3 in (4.9) gives

$$Nu_5 = A_9 \cos x + A_{10} \cos 3x + A_{11} \cos 5x \tag{4.16a}$$

where,

$$A_9 = \left(\frac{11\beta\bar{b}^5}{16} + 2\gamma_5 - \frac{3\alpha\bar{b}^5}{128}\right) \tag{4.16b}$$

$$A_{10} = \left(\frac{\beta\bar{b}^5}{4} - \frac{9\alpha\bar{b}^3}{32} - \frac{3\alpha\bar{b}^5}{64}\right), \quad A_{11} = \left(\frac{\beta\bar{b}^5}{16} - \frac{3\alpha\bar{b}^5}{128}\right) \tag{4.16c}$$

On account of (3.7), we observe that $A_9 = 0$. This yields

$$\gamma_5 = \frac{1}{2}\left(\frac{3\alpha\bar{b}^5}{128} - \frac{11\beta\bar{b}^5}{16}\right) \tag{4.17a}$$

The remaining equation in (4.16a) is solved to get

$$u_5 = -\left(\frac{A_{10}\cos 3x}{8} + \frac{A_{11}\cos 5x}{24}\right) \tag{4.17b}$$

On substituting for relevant terms in (4.10), we obtain

$$\gamma_6 = 0, \qquad u_6 = 0 \tag{4.18}$$

After substituting for terms in (4.11) and simplifying, the equation becomes

$$Nu_7 = A_{12}\cos x + A_{13}\cos 3x + A_{14}\cos 5x + A_{15}\cos 7x \tag{4.19}$$

where,

$$A_{12} = \left[\frac{15\alpha\beta\bar{b}^7}{512} + 2\gamma_7 + \alpha\left\{\frac{3\bar{b}^2A_9}{16} - \frac{3\alpha^2\bar{b}^7}{2048}\right\}\right] \tag{4.20a}$$

$$A_{13} = \left[\frac{9A_9}{8} + \alpha\left\{\frac{3\bar{b}^2A_9}{16} + \frac{3\bar{b}A_{11}}{48}\right\} + \frac{15\alpha\beta\bar{b}^7}{256}\right] \tag{4.20b}$$

$$A_{14} = \left[\frac{25A_{11}}{4} + \alpha\left\{\frac{3\bar{b}^2A_{11}}{48} + \frac{3\bar{b}^2A_9}{16} - \frac{3\alpha^2\bar{b}^7}{1024}\right\} + \frac{5\alpha\beta\bar{b}^7}{256}\right] \tag{4.20c}$$

$$A_{15} = \left[\left\{\frac{3\alpha\bar{b}A_{11}}{48} - \frac{3\alpha^3\bar{b}^7}{4096}\right\} + \frac{5\alpha\beta\bar{b}^7}{512}\right] \tag{4.20d}$$

From the orthogonality condition (3.7) as applied to (4.19), we get

$$\gamma_7 = -\frac{1}{2}\left[\frac{15\alpha\beta\bar{b}^7}{512} + \left\{\frac{3\bar{b}^2A_9}{16} - \frac{3\alpha^2\bar{b}^7}{2048}\right\}\right] \tag{4.21}$$

The solution of the remaining equation in (4.19) is

$$u_7 = -\frac{1}{2}\left[\frac{A_{13}\cos 3x}{(41-9\lambda)} + \frac{A_{14}\cos 5x}{(313-25\lambda)} + \frac{A_{15}\cos 7x}{(1201-49\lambda)}\right] \tag{4.22}$$

From (4.3) and (4.4), we write

$$w(x) = \bar{b}\varepsilon + \frac{\varepsilon^3\alpha\bar{b}^3\cos 3x}{32} - \varepsilon^5\left(\frac{A_{10}\cos 3x}{8} + \frac{A_{11}\cos 5x}{24}\right) - \frac{\varepsilon^7}{2}\left[\frac{A_{13}\cos 3x}{(41-9\lambda)} + \frac{A_{14}\cos 5x}{(313-25\lambda)} + \frac{A_{15}\cos 7x}{(1201-49\lambda)}\right] + \cdots \tag{4.23}$$

Similarly, we have, from (4.4),

$$\lambda\epsilon = \frac{\bar{b}\varepsilon^3}{2}\left(1+\frac{3\alpha\bar{b}^2}{4}\right) + \frac{\bar{b}^5\varepsilon^5}{2}\left(\frac{3\alpha}{128} - \frac{11\beta}{16}\right) - \frac{\bar{b}^2\varepsilon^7}{2}\left[\frac{15\alpha\beta\bar{b}^5}{512} + \alpha\left\{\frac{3A_9}{16} - \frac{3\alpha^2\bar{b}^5}{2048}\right\}\right] + \cdots \tag{4.24}$$

To determine the buckling load λ_S, we employ (3.30), which yields

$$\frac{3\bar{b}\varepsilon_S^2}{2}\left(1+\frac{3\alpha\bar{b}^2}{4}\right) + \frac{5\bar{b}^5\varepsilon_S^4}{2}\left(\frac{3\alpha}{128} - \frac{11\beta}{16}\right) - \frac{7\bar{b}^2\varepsilon_S^6}{2}\left[\frac{15\alpha\beta\bar{b}^5}{512} + \alpha\left\{\frac{3A_9}{16} - \frac{3\alpha^2\bar{b}^5}{2048}\right\}\right] = 0 \tag{4.25}$$

At this stage, we shall give the result in two levels of approximation. First, if we take only the first two terms in (4.25), we get

$$\frac{3\bar{b}\varepsilon_S^2}{2}\left(1+\frac{3\alpha\bar{b}^2}{4}\right) + \frac{5\bar{b}^5\varepsilon_S^4}{2}\left(\frac{3\alpha}{128} - \frac{11\beta}{16}\right) = 0 \tag{4.26a}$$

where ε_S is the value of ε at static buckling. This gives

$$\varepsilon_S^2 = \frac{3}{5\bar{b}^4}\left(\frac{1+\frac{3\alpha\bar{b}^2}{4}}{\frac{11\beta}{16} - \frac{3\alpha}{128}}\right), \qquad \varepsilon_S = \frac{1}{\bar{b}^2}\sqrt{\frac{3}{5}}\left(\frac{1+\frac{3\alpha\bar{b}^2}{4}}{\frac{11\beta}{16} - \frac{3\alpha}{128}}\right)^{\frac{1}{2}} \tag{4.26b}$$

Now, on evaluating (4.24) at buckling, where $\lambda = \lambda_S$, we get

$$\lambda_S\epsilon = \frac{1}{2\bar{b}^5}\left(\frac{3}{5}\right)^{\frac{3}{2}}\left(\frac{1+\frac{3\alpha\bar{b}^2}{4}}{\frac{11\beta}{16} - \frac{3\alpha}{128}}\right)^{\frac{1}{2}}\left[\left(1+\frac{3\alpha\bar{b}^2}{4}\right) - \varepsilon_S^2\left\{\bar{b}^4\left(\frac{11\beta}{16} - \frac{3\alpha}{128}\right) + \varepsilon_S^2 A_{16}\right\}\right] \tag{4.27a}$$

where,

$$A_{16}(\lambda_S) = \left[\frac{15\alpha\beta\bar{b}^5}{512} + \alpha\left\{\frac{3A_9}{16} - \frac{3\alpha^2\bar{b}^5}{2048}\right\}\right] \tag{4.27b}$$

and where (4.27a, b) are evaluated at where $\lambda = \lambda_S$. If we take the three terms in (4.25) then, we can write the whole equation as

$$\varepsilon_S^2\left[\frac{3\bar{b}}{2}A_{17} + \frac{5\bar{b}^4\varepsilon_S^2}{2}A_{18} - \frac{7\bar{b}^2\varepsilon_S^4}{2}A_{19}\right] = 0 \tag{4.28a}$$

where,

$$A_{17} = \left(1 + \frac{3\alpha\bar{b}^2}{4}\right), \quad A_{18} = -\left(\frac{11\beta}{16} - \frac{3\alpha}{128}\right), \qquad A_{19} = \left[\frac{15\alpha\beta\bar{b}^5}{512} + \alpha\left\{\frac{3A_9}{16} - \frac{3\alpha^2\bar{b}^5}{2048}\right\}\right] \qquad (4.28b)$$

Then, we can recast (4.28a) simply as

$$C_1\varepsilon_S^4 - C_2\varepsilon_S^2 - C_3 = 0 \qquad (4.29a)$$

where,

$$C_1 = \frac{7\bar{b}^2}{2}A_{19}, \quad C_2 = \frac{5\bar{b}^4}{2}A_{18}, \qquad C_3 = \frac{3\bar{b}}{2}A_{17} \qquad (4.29b)$$

The solution of (4.29a) is

$$\varepsilon_S^2 = \frac{5\bar{b}^2 A_{18}}{7A_{19}}\left[1 - \sqrt{1 + \frac{84A_{17}A_{19}}{25\bar{b}^6 A_{18}^2}}\right] \qquad (4.30a)$$

$$\varepsilon_S = \bar{b}\sqrt{\frac{5}{7}}\left(\frac{A_{18}}{A_{19}}\right)^{\frac{1}{2}}\left[1 - \sqrt{1 + \frac{84A_{17}A_{19}}{25\bar{b}^6 A_{18}^2}}\right]^{\frac{1}{2}} \qquad (4.30b)$$

The static buckling load in this case is determined using (4.24) at $\lambda = \lambda_S$ and using the values of ε_S^2 and ε_S as in (4.30a, b) respectively. This gives

$$\lambda_S\epsilon = \frac{\bar{b}\varepsilon_S^3}{2}\left[\left(1 + \frac{3\alpha\bar{b}^2}{4}\right) - \frac{\varepsilon_S^2}{2}\left\{\left(\left(\frac{11\beta}{16} - \frac{3\alpha}{128}\right) + \bar{b}\varepsilon_S^2\left\{\frac{15\alpha\beta\bar{b}^5}{512} + \alpha\left\{\frac{3A_9}{16} - \frac{3\alpha^2\bar{b}^5}{2048}\right\}\right\}\right)\right\}\right] \qquad (4.31)$$

5 Analysis and Discussion of Results

The results (3.34), (4.27a) and (4.31) show mathematical relationship between the Static buckling load λ_S and the imperfection parameter ϵ. Using Q – Basic codes with $\bar{b} = 0.5$, the results obtained from the two methods are shown both on Table1 and Table2 as well as on Figure1 and Figure2. It is clearly shown that the Static buckling load, in each case, decreases with increased imperfection parameter. All results are implicit in the load parameter λ_S and are valid provided the perturbation parameters are small relative to unity. It is pertinent that $\bar{b}$ satisfies the inequality $0 < \bar{b} < 1$. Certainly, if the values of α, β and $\bar{b}$ change, a new set of results will be obtained. But whatever be the values of these parameters within the limits allowable in this work, the general trend will be the same, namely the static buckling load λ_S decreases with increase in imperfection ϵ and vice versa. This trend is characteristic of all imperfection sensitive structures. In particular, it is observed that as the imperfection tends to zero, λ_S asymptotically increases without bounds and as it tends to one, λ_S tends to approach zero.

Though we have limited our analysis to $0 < \epsilon \ll 1$, the same trend follows if we use $-1 \ll \epsilon < 0$.

5.1 Numerical and graphical plots

Below, we give the numerical results and the graphical plots of the relationship between the static buckling load and the imperfection parameter using the two methods.

Table 1. Relationship between the Static buckling load λ_S and the Imperfection parameter, ϵ for α = 1, β = 1 using Eqn. (3.34)

Imperfection parameter, ϵ	λ_S for α = 1, β = 1
0.01	0.286212
0.02	0.285966
0.03	0.285721
0.04	0.285478
0.05	0.285236
0.06	0.284995
0.07	0.284756
0.08	0.284519
0.09	0.284283
0.1	0.284048

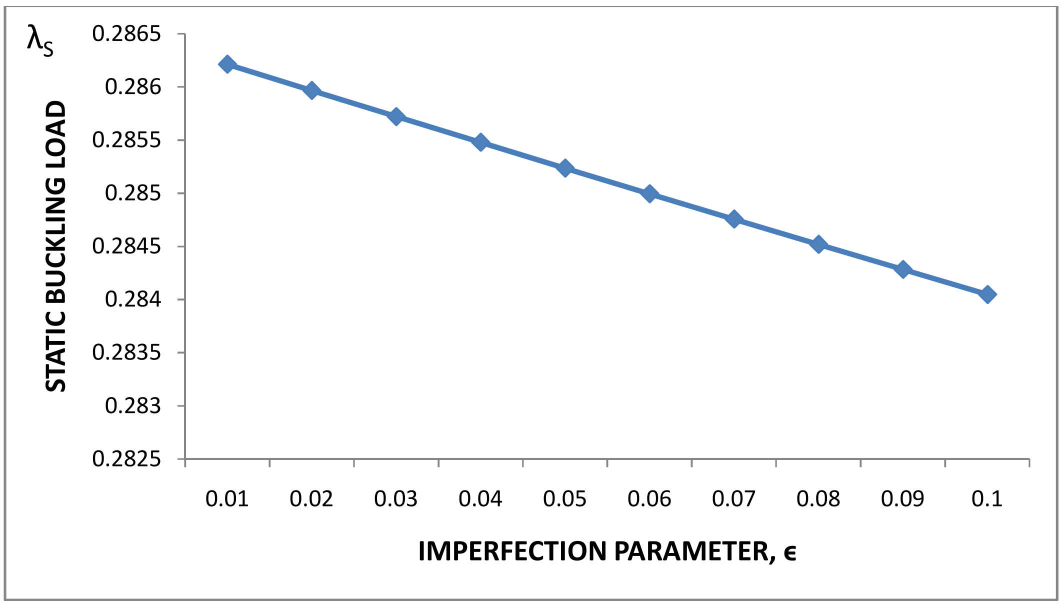

Fig. 1. Graphical Plot of Table 1, showing the relationship between the Static buckling load λ_S and the Imperfection parameter, ϵ for α = 1, β = 1, using Eqn. (3.34)

Table 2. Relationship between the Static buckling load λ_S and the Imperfection parameter, ϵ for α = 1, β = 1 and $\bar{b} = 0.5$, using Eqn. (4.27a)

Imperfection parameter, ϵ	λ_S for α = 1, β = 1, $\bar{b} = 0.5$
0.01	0.571931
0.02	0.285966
0.03	0.190644
0.04	0.142983
0.05	0.114387
0.06	0.095322
0.07	0.081705
0.08	0.071492
0.09	0.063548
0.1	0.057194

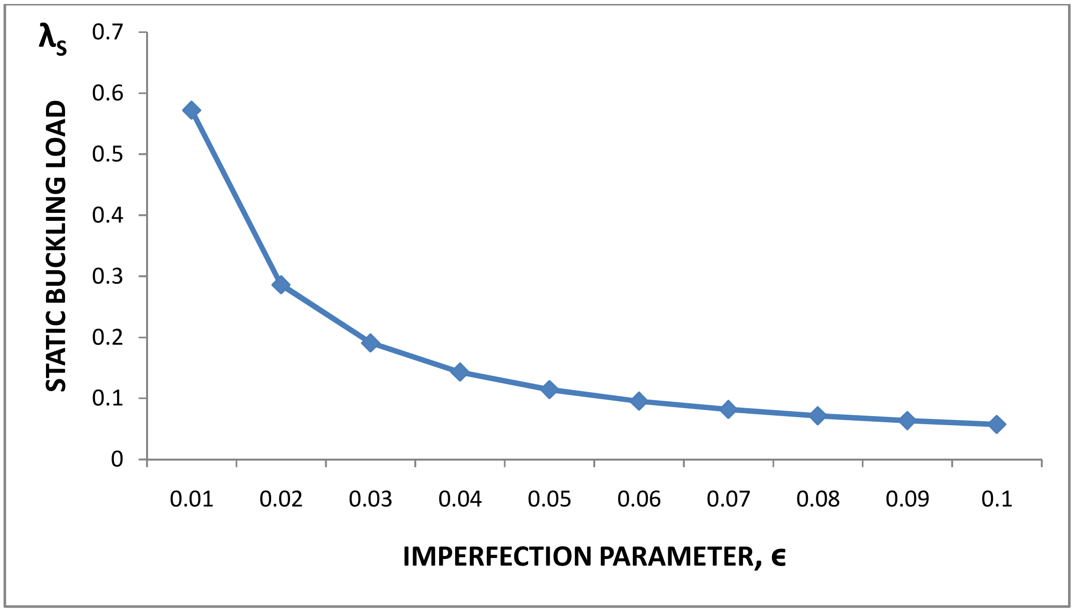

Fig. 2. Graphical Plot of Table 2, showing the relationship between the Static buckling load λ_S and the Imperfection parameter, ϵ for α = 1, β = 1 and $\bar{b} = 0.5$, using Eqn. (4.27a)

6 Conclusion

The paper has used perturbation methods to analyze a problem in non – linear dynamical system. The results are asymptotic and so, are valid as long as the small parameters are small relative to unity. Moreso, the result is implicit in the load parameter. The same method and technique can be used to analyze structurally more complex materials like cylindrical shells and plates subjected to similar loading conditions.

Competing Interests

Authors have declared that no competing interests exist.

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