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Fuzzy Relational Equations of k - regular Intuitionistic Fuzzy and Block Fuzzy Matrices

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Authors' contributions

This work was carried out in collaboration between both authors. Author PJ designed the study. Author EK managed the analyses of the study. Both authors read and approved the final manuscript.

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ABSTRACT

In this paper, the solution of fuzzy relational equations are determined in the case of k - regular intuitionistic fuzzy matrices. Also we introduce the concept of k - regularity for block intuitionistic fuzzy matrices and the consistency of intuitionistic fuzzy relational equations are discussed.

Keywords: Intuitionistic fuzzy matrices (IFMs); k - regular intuitionistic fuzzy matrix; k - regular intuitionistic block fuzzy matrix; fuzzy relational equation; k - g inverse.

1. INTRODUCTION

Let F_n be the set of all $n \times n$ fuzzy matrices over the fuzzy algebra $F = \{0,1\}$ under the operations $(+,\cdot)$ defined as $a + b = max\{a, b\}$ and $a \cdot b = min\{a, b\}$ for all $a, b \in F$. In short F_n denotes fuzzy marices of order $n \times n$. Kim and Roush [1] have given a systematic development of fuzzy matrix theory, introducing new definitions such as the independence of a set of fuzzy vectors defined over a commutative semiring, the basis of a subspace of fuzzy vectors, the Schein rank of an $n \times n$ fuzzy matrix A, etc. Of course, these definitions are the generalizations of similar

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definitions in the setting of Boolean matrix theory [2]. For $A \in F_{mn}, R(A)$ and $C(A)$ denote the row space and column space respectively. $A \in F_{mn}$ is said to be regular if there exist X such that $AXA = A$, X is called the generalized inverse of A. $A\{1\}$ denotes the set of all g-inverses of A. Meenakshi and Jenita have extended the notion of regular matrices to k - regular matrices for some positive integer k [3]. Atanassov has introduced and developed the concept of intuitionistic fuzzy sets as a generalization of fuzzy sets [4]. Pal, Khan and Shyamal have been studied the concept of intuitionistic fuzzy matrices [5]. A study on regularity and various g - inverse of intuitionistic fuzzy matrices over intuitionistic fuzzy algebra are discussed in [6]. Basic properties of intuitionistic fuzzy matrices as a generalization of the results on fuzzy matrices have been derived by Khan and Anita Paul [7]. After Sanchez [8] introduced the fuzzy relation equations, several authors have further enlarged this theory with many papers. In [9], Cho has discussed the consistency of fuzzy matrix equations. Zhou Wei and boa Menghong extended the fuzzy relational equations to intuitionistic fuzzy relational equations [10]. A necessary and sufficient condition for the fuzzy relation equations are found in [11, 12]. An applicability of the numerical solutions of the fuzzy systems are discussed in [13]. Regularity of block fuzzy matrices and its properties are discussed in [14]. The concept of regularity for block intuitionistic fuzzy matrices and consistency of intuitionistic fuzzy relational equations are discussed by Meenakshi and Gandhimathi [15]. Further to learn about fuzzy sets, fuzzy matrix theory and its applications, one may refer [16, 17]. The solutions of fuzzy relational equations are determined in the case of k - regular fuzzy matrices and block fuzzy matrices are discussed by Jenita [18, 19]. A sufficient condition for existence of the smallest solution of a max-min fuzzy equations found in [20]. In [21], Higashi and Klir have derived the general schemes for solving fuzzy relation equations with finite sets. Fuzzy relation equations with triangular norms and their resolution are discussed in [22]. In [23], Di Nola and Sessa introduced some algorithms which have minimization properties about the fuzziness of solutions in the maxmin fuzzy relation equations. Applications of fuzzy relation equations are discussed in [24-27]. Applications of fuzzy models and its procedures have been discussed in [28]. Approximate solutions of fuzzy relation equations are found in [29,30]. Solutions of fuzzy relation equations with extended operations introduced in [31]. Recently, we have introduced the concept of k-regular intuitionistic fuzzy matrix as a generalization of regular intuitionistic fuzzy matrix [32]. Further to learn about fuzzy relation equation, one may refer [33,34]. In this paper, the solution of fuzzy relational equations are determined in the case of k - regular intuitionistic fuzzy and block fuzzy matrices.

2. PRELIMINARIES

Here, we are concerned with fuzzy matrices, that is matrices over a fuzzy algebra FM(FN) with support $[0,1]$, under maxmin(minmax) operations and the usual ordering of real numbers. Let $(IF)_{m\times n}$ be the set of all intuitionistic fuzzy matrices of order $m \times n$, $F^M_{m\times n}$ be the set of all fuzzy matrices of order $m \times n$, under the maxmin composition and $F^N_{m\times n}$ be the set of all fuzzy matrices of order $m \times n$, under the minmax composition. In short $(IF)_n$ denotes the intuitionistic fuzzy matrix of order $n \times n$.

If $A = (a_{ij}) \in (IF)_{m\times n}$, then $A = \left(\langle a_{ij\mu}, a_{ij\vartheta}\rangle\right)$, where $a_{ij\mu}$ and $a_{ij\vartheta}$ are the membership values and non membership values of a_{ij} in A respectively with respect to the fuzzy sets μ and ϑ, maintaining the condition $0 \le a_{ij\mu} + a_{ij\vartheta} \le 1$.

We shall follow the matrix operations on intuitionistic fuzzy matrices as defined in [20].
For $A, B \in (IF)_{m\times n}$, then

$$A + B = \left(\langle \max\{a_{ij\mu}, b_{ij\mu}\}, \min\{a_{ij\vartheta}, b_{ij\vartheta}\}\rangle\right)$$
$$AB = \left(\left\langle \max_k \min\{a_{ik\mu}, b_{kj\mu}\}, \min_k \max\{a_{ik\vartheta}, b_{kj\vartheta}\}\right\rangle\right)$$

Let us define the order relation on $(IF)_{m\times n}$ as,

$A \le B \Leftrightarrow a_{ij\mu} \le b_{ij\mu}$ and $a_{ij\vartheta} \ge b_{ij\vartheta}$, for all i and j.

In this work, we shall represent $A \in (IF)_{m\times n}$ as cartesian product of fuzzy matrices.

For $A = (a_{ij}) \in (IF)_{m\times n}$. Let $A = (a_{ij}) = \left(\langle a_{ij\mu}, a_{ij\vartheta}\rangle\right) \in (IF)_{m\times n}$.

We define $A_\mu = (a_{ij\mu}) \in F^M_{m\times n}$ as the membership part of A and $A_\vartheta = (a_{ij\vartheta}) \in F^N_{m\times n}$ as the non-membership part of A. Thus A is written as the cartesian product of A_μ and A_ϑ, $A = < A_\mu, A_\vartheta >$ with $A_\mu \in F^M_{m\times n}, A_\vartheta \in F^N_{m\times n}$. For

$A \in (IF)_{m\times n}, R(A)(C(A))$ be the space generated by the rows (columns) of A.

Definition 2.1. [6]

For $A, B \in (IF)_{m\times n}$, if $A =< A_\mu, A_\vartheta >$ and $B =< B_\mu, B_\vartheta >$,

then $A + B =< A_\mu + B_\mu, A_\vartheta + B_\vartheta >$.

Definition 2.2. [6]

For $A \in (IF)_{m\times p}, B \in (IF)_{p\times n}$ if $A =< A_\mu, A_\vartheta >$ and $B =< B_\mu, B_\vartheta >$, then

(i) $AB =< A_\mu B_\mu, A_\vartheta B_\vartheta >$, where $A_\mu B_\mu$ is the maxmin product in $F^M_{m\times n}$ and $A_\vartheta B_\vartheta$ is the minmax product in $F^N_{m\times n}$.
(ii) $A^T =< A^T_\mu, A^T_\vartheta >$.

Definition 2.3. [32]

A matrix $A \in (IF)_n$, is said be right k-regular if there exists a matrix $X \in (IF)_n$ such that $A^k XA = A^k$, for some positive integer k. X is called a right k-g-inverse of A.

Let $A_r\{1^k\} = \{X/A^k XA = A^k\}$.

Definition 2.4. [32]

A matrix $A \in (IF)_n$, is said be left k-regular if there exists a matrix $Y \in (IF)_n$ such that $AYA^k = A^k$, for some positive integer k. Y is called a left k-g-inverse of A.

Let $A_\ell\{1^k\} = \{Y/AYA^k = A^k\}$. Let $A\{1^k\}$ be the set of k-g-inverses of A.

Lemma 2.5. [6]

For $A, B \in (IF)_{m\times n}, R(B) \subseteq R(A) \Leftrightarrow B = XA$ for some $X \in (IF)_m, C(B) \subseteq C(A) \Leftrightarrow B = AY$ for some $Y \in (IF)_n$.

Lemma 2.6. [17]

If $A \in (IF)_{m\times n}$ is of the form $A =< A_\mu, A_\vartheta >$, then

(i) $R(A) =< R(A_\mu), R(A_\vartheta) >$ and
(ii) $C(A) =< C(A_\mu), C(A_\vartheta) >$.

Theorem 2.7. [32]

Let $A = \langle A_\mu, A_\vartheta \rangle \in (IF)_n$. Then A is right(left) k-regular IFM $\Leftrightarrow A_\mu, A_\vartheta \in F_n$ are right(left) k-regular.

Theorem 2.8. [6]

Let $A \in (IF)_{m\times n}$ be of the form $A = \langle A_\mu, A_\vartheta \rangle$. Then A is regular $\Leftrightarrow A_\mu$ is regular in $F^M_{m\times n}$ under max-min composition and A_ϑ is regular in $F^N_{m\times n}$ under min-max composition. $A_\mu = (a_{ij\mu}) \in F^M_{m\times n}$ as the membership part of A and $A_\vartheta = (a_{ij\vartheta}) \in F^N_{m\times n}$ as the non-membership part of A.

3. FUZZY RELATIONAL EQUATIONS OF K - REGULAR INTUITIONISTIC FUZZY MATRICES

In this section, the solution of fuzzy relational equations are determined in the case of k - regular intuitionistic fuzzy matrices.

Lemma 3.1.

For $A, B \in (IF)_n$, and a positive integer k, then

(i) If A is right k - regular and $R(B) \subseteq R(A^k)$ then $B = BXA$ for each right k - g inverse X of A.
(ii) If A is left k - regular and $C(B) \subseteq C(A^k)$ then $B = AYB$ for each left k - g inverse Y of A.

Proof:

(i) Since $R(B) \subseteq R(A^k)$, by Lemma (2.5), there exists Y such that $B = YA^k$.

By Definition (2.3), $A^k XA = A^k$.

Hence $B = YA^k = YA^k XA = (YA^k)XA = BXA$.

Thus (i) holds.

(ii) This can be proved in the same manner.

Theorem 3.2.

For $A, B, D \in (IF)_n$ and $Y \in A\{1^k_\ell\}, Z \in B\{1\}$. If the intuitionistic fuzzy matrix equation $A^k XB = D$ is solvable then $AYDZB = D$.

Proof:

Let X be any solution of $A^k XB = D$.

$$\begin{aligned} D &= A^k XB \\ &= AYA^k XBZB \\ &= AY(A^k XB)ZB \\ &= AYDZB. \end{aligned}$$

Hence the proof.

Theorem 3.3.

For $A, B, D \in (IF)_n$ and $Y \in A\{1\}, Z \in B\{1_r^k\}$. If the intuitionistic fuzzy matrix equation $AXB^k = D$ is solvable then $AYDZB = D$.

Proof:

Let X be any solution of $AXB^k = D$.

$$\begin{aligned} D &= AXB^k \\ &= AYAXB^kZB \\ &= AY(AXB^k)ZB \\ &= AYDZB. \end{aligned}$$

Hence the proof.

Theorem 3.4.

For $A, B, D \in (IF)_n$ and $Y \in A\{1_\ell^k\}$ and $Z \in B\{1_r^k\}$. If the intuitionistic fuzzy matrix equation $A^kXB^k = D$ is solvable then $AYDZB = D$.

Proof:

Let X be any solution of $A^kXB^k = D$.

$$\begin{aligned} D &= A^kXB^k \\ &= AYA^kXB^kZB \\ &= AY(A^kXB^k)ZB \\ &= AYDZB. \end{aligned}$$

Hence the proof.

Remark 3.5.

For $k = 1$, the Theorem (3.2) to (3.4) reduces to the following Theorem:

Theorem 3.6.

Let $A = \langle A_\mu, A_\vartheta \rangle \in (IF)_{m\times n}, B = \langle B_\mu, B_\vartheta \rangle \in (IF)_{p\times q}$ be regular IFMs and $D = \langle D_\mu, D_\vartheta \rangle \in (IF)_{m\times q}$. Thus the intuitionistic fuzzy matrix equation $AXB = D$ is solvable iff $AA^-DB^-B = D$ for $A^- \in A\{1\}$ and $B^- \in B\{1\}$.

Remark 3.7.

Theorem (3.6) is a generalization of the following theorem.

Theorem 3.8. [17]

Let $A, B \in (IF)_n$ be a regular IFMs and $D \in (IF)_n$. Then the intuitionistic fuzzy matrix equation $AXB = D$ is solvable iff $AYDZB = D$ for $Y \in A\{1\}$ and $Z \in B\{1\}$.

4. FUZZY RELATIONAL EQUATIONS OF K - REGULAR BLOCK INTUITIONISTIC FUZZY MATRICES

In [15], Gandhimathi and Meenakshi have introduced, the Schur complements in block intuitionistic fuzzy matrix as an extension of fuzzy matrices found in [14].

In this section, we are concerned with a block intuitionistic fuzzy matrix of the form

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{4.1}$$

with the diagonal block A and D are k - regular IFM with respect to this partitioning a Schur complement of A in M is a matrix of the form $M/A = D - CXB$, where X is some k - g inverse of A . Similarly $M/D = A - BYC$ is a Schur complement of D in M, where Y is some k - g inverse of D. In Theorem [4.1] , it is shown that under certain conditions CXB is invariant for all choices of k - g inverse X of A. By M/A is an intuitionistic fuzzy matrix, we mean that CXB is invariant and $D \geq CXB$. Therefore

M/A is an intuitionistic fuzzy matrix $\Leftrightarrow$ CXB is invariant and $D = D + CXB$ (4.2)

Similarly,

$M/D = A - BYC$ is an intuitionistic fuzzy matrix $\Leftrightarrow$ BYC is invariant and $A = A + BYC$ (4.3)

Let M be of the form (4.1) can be expressed as $M = \langle M_\mu, M_\vartheta \rangle$, where $M_\mu = \begin{bmatrix} A_\mu & B_\mu \\ C_\mu & D_\mu \end{bmatrix}$ and $M_\vartheta = \begin{bmatrix} A_\vartheta & B_\vartheta \\ C_\vartheta & D_\vartheta \end{bmatrix}$ are block IFM. $A = \langle A_\mu, A_\vartheta \rangle, B = \langle B_\mu, B_\vartheta \rangle, C = \langle C_\mu, C_\vartheta \rangle$ and $D = \langle D_\mu, D_\vartheta \rangle$. Since A and D are k - regular, by Theorem [2.7], $A_\mu, A_\vartheta, D_\mu$ and D_ϑ are all k-regular IFMs.

Theorem 4.1.

Let $A \in (IF)_n$ be a k-regular intuitionistic fuzzy matrix, $C \in (IF)_n$ and $B \in (IF)_n$ if $R(C) \subseteq R(A^k)$ and $C(B) \subseteq C(A^k)$ Then CXB is invariant for all choice of k-g inverses of A.

Proof:

Case (i): A is right k-regular.

By Lemma [2.5], $R(C) \subseteq R(A^k) \Rightarrow C = YA^k$ for some $Y \in (IF)_n$ and
$C(B) \subseteq C(A^k) \subseteq C(A) \Rightarrow B = AU$ for some $U \in (IF)_n$.

Since $A \in (IF)_n$ is a right k-regular intuitionistic fuzzy matrix by Lemma [3.1],

$R(C) \subseteq R(A^k) \Rightarrow C = CZA$ for each $Z \in A\{1_r^k\}$.

Hence for any $X \in A\{1_r^k\}$,

$$\begin{aligned} CXB &= (YA^k)X(AU) = YA^kXAU = Y(A^kXA)U \\ &= YA^kU = CU = CZAU = CZ(AU) \\ &= CZB \end{aligned}$$

Thus $CXB = CZB$ for all $X, Z \in A\{1_r^k\}$.

Case (ii): A is left k-regular.

By Lemma [2.5], $R(C) \subseteq R(A^k) \subseteq R(A) \Rightarrow C = YA$ for some $Y \in (IF)_n$ and
$C(B) \subseteq C(A^k) \Rightarrow B = A^kU$ for some $U \in (IF)_n$.

Since $A \in (IF)_n$ is a left k - regular intuitionistic fuzzy matrix, by Lemma [3.1],

$C(B) \subseteq C(A^k) \Rightarrow B = AZB$ for each $Z \in A\{1_\ell^k\}$.

Hence for any $X \in A\{1_\ell^k\}$,

$$\begin{aligned} CXB &= (YA)X(A^kU) = Y(AXA^k)U = YA^kU = YB \\ &= Y(AZB) = (YA)(ZB) = CZB. \end{aligned}$$

Thus $CXB = CZB$ for all $X, Z \in A\{1_\ell^k\}$.

Case (iii): A is both right and left k-regular.

By Lemma [2.5], $R(C) \subseteq R(A^k) \Rightarrow C = YA^k$ for some $Y \in (IF)_n$.

Since $A \in (IF)_n$ is a left k-regular intuitionistic fuzzy matrix, by Lemma [3.1],

$C(B) \subseteq C(A^k) \Rightarrow B = AZB$ for each $Z \in A\{1_\ell^k\}$.

Since $A \in (IF)_n$ is a right k-regular intuitionistic fuzzy matrix, for any $X \in A\{1_r^k\}$,

$CXB = (YA^k)X(AZB) = Y(A^kXA)ZB = YA^kZB = CZB$.

Thus $CXB = CZB$ for all $X \in A\{1_r^k\}$ and $Z \in A\{1_\ell^k\}$.

Thus CXB is invariant for all choices of k - g inverses of A.

Theorem 4.2.

Let $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ with A and D are right k - regular intuitionistic fuzzy matrices, M/A and M/D are exists. $R(C) \subseteq R(A^k)$ and $R(B) \subseteq R(D^k)$. If $xM = b$ is solvable then $yA = c$ and $zD = d$ are solvable, where $b = \begin{pmatrix} c & d \end{pmatrix}$, $c \geq dD^-C$ and $d \geq cA^-B$.

Proof:

Since $xM = b$ is solvable, let $x = \begin{pmatrix} \beta & \gamma \end{pmatrix}$ is a solution.

Then, $\begin{pmatrix} \beta & \gamma \end{pmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{pmatrix} c & d \end{pmatrix} \Rightarrow$
$\begin{pmatrix} \beta A + \gamma C & \beta B + \gamma D \end{pmatrix} = \begin{pmatrix} c & d \end{pmatrix}$

Hence we get the equations,

$$\beta A + \gamma C = c \text{ and } \beta B + \gamma D = d. \qquad (4.4)$$

By Lemma [3.1], A is right k-regular intuitionistic fuzzy matrices, $R(C) \subseteq R(A^k) \Rightarrow C = CA^-A$ for each right k - g - inverse A^- of A and D is right k-regular intuitionistic fuzzy matrix, $R(B) \subseteq R(D^k) \Rightarrow B = BD^-D$ for each right k-g inverse D^- of D.

Substituting C and B in Equation (4.4), we get the equations,

$(\beta + \gamma CA^-)A = c$ and $(\beta BD^- + \gamma)D = d$.

Thus $yA = c$ and $zD = d$ are solvable. Since A and D are right k-regular intuitionistic fuzzy matrices, the solutions will be of the form $y = cA^-$ and $z = dD^-$.
Hence $cA^- = \beta + \gamma CA^-$ and $dD^- = \beta BD^- + \gamma$.

$$cA^-B = \beta B + \gamma CA^-B \text{ and } dD^-C = \beta BD^-C + \gamma C \qquad (4.5)$$

Since M/A and M/D exist then $A + BD^-C = A$ and $D + CA^-B = D$.

Substituting for A and D in (4.4) using (4.5) we get

$$c = \beta A + \gamma C = \beta A + \beta B D^{-} C + \gamma C = \beta A + d D^{-} C$$
$$d = \beta B + \gamma D = \beta B + \gamma D + \gamma C A^{-} B = \gamma D + c A^{-} B.$$

By intuitionistic fuzzy addition it follows that $c \geq d D^{-} C$ and $d \geq c A^{-} B$.

This is illustrated in the following example.

Example 4.3.

Let $M = \begin{bmatrix} \langle 0.3,0\rangle & \langle 0,1\rangle & \vdots & \langle 0.2,0.4\rangle & \langle 0.1,0.4\rangle \\ \langle 0.5,0\rangle & \langle 0.2,0\rangle & \vdots & \langle 0.2,0.3\rangle & \langle 0.2,0.3\rangle \\ \cdots\cdots & \cdots\cdots & \cdots & \cdots\cdots & \cdots\cdots \\ \langle 0.2,0.2\rangle & \langle 0.1,0.2\rangle & \vdots & \langle 0.2,0.3\rangle & \langle 0.1,0\rangle \\ \langle 0,0.2\rangle & \langle 0,0.2\rangle & \vdots & \langle 0.4,0\rangle & \langle 0.2,0\rangle \end{bmatrix}$,

where $A = \begin{bmatrix} \langle 0.3,0\rangle & \langle 0,1\rangle \\ \langle 0.5,0\rangle & \langle 0.2,0\rangle \end{bmatrix}$, $B = \begin{bmatrix} \langle 0.2,0.4\rangle & \langle 0.1,0.4\rangle \\ \langle 0.2,0.3\rangle & \langle 0.2,0.3\rangle \end{bmatrix}$,

$C = \begin{bmatrix} \langle 0.2,0.2\rangle & \langle 0.1,0.2\rangle \\ \langle 0,0.2\rangle & \langle 0,0.2\rangle \end{bmatrix}$ and $D = \begin{bmatrix} \langle 0.2,0.3\rangle & \langle 0.1,0\rangle \\ \langle 0.4,0\rangle & \langle 0.2,0\rangle \end{bmatrix}$.

$A = \langle A_\mu, A_\vartheta\rangle, B = \langle B_\mu, B_\vartheta\rangle, C = \langle C_\mu, C_\vartheta\rangle$ and $D = \langle D_\mu, D_\vartheta\rangle$.

To prove that A is not regular.

$A = \begin{bmatrix} \langle 0.3,0\rangle & \langle 0,1\rangle \\ \langle 0.5,0\rangle & \langle 0.2,0\rangle \end{bmatrix} \in (IF)_2$, where $A_\mu = \begin{bmatrix} 0.3 & 0 \\ 0.5 & 0.2 \end{bmatrix} \in F_2^M$ and $A_\vartheta = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in F_2^N$. Since each row of A_μ cannot be expressed as linear combination of the other row, by Definition 2.5 of (1), the rows are linearly independent. By Definition 2.6 of (9) they form a standard basis for the row space of A_μ. For both permutation matrices $P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $P_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $A_\mu P_1 A_\mu = \begin{bmatrix} 0.3 & 0 \\ 0.3 & 0.2 \end{bmatrix} \neq A_\mu$ and $A_\mu P_2 A_\mu = \begin{bmatrix} 0.3 & 0.2 \\ 0.5 & 0.2 \end{bmatrix} \neq A_\mu$. Hence A_μ is not regular by step 3 in Algorithm 1 of (9). Namely, A_μ is regular iff $A_\mu P A_\mu = A_\mu$ for some permutation matrix P. Since A_ϑ is idempotent, A_ϑ itself is a g-inverse of A_ϑ, therefore A_ϑ is regular under min max composition. Hence by Theorem 2.8, A is not regular.

For this A, $A^2 = \begin{bmatrix} \langle 0.3,0\rangle & \langle 0,1\rangle \\ \langle 0.3,0\rangle & \langle 0.2,0\rangle \end{bmatrix}$. For $X = \begin{bmatrix} \langle 1,0\rangle & \langle 0,1\rangle \\ \langle 0,0\rangle & \langle 0.2,0\rangle \end{bmatrix}$, $A^2 X A = A^2 = A X A^2$ holds.

Hence A is 2-regular.

Similarly, we can prove that, D is not regular. For this D, $D^2 = \begin{bmatrix} \langle 0.2,0.3\rangle & \langle 0.1,0\rangle \\ \langle 0.2,0\rangle & \langle 0.2,0\rangle \end{bmatrix}$.

For $Y = \begin{bmatrix} \langle 0.2,0.3\rangle & \langle 0.1,0\rangle \\ \langle 0,0\rangle & \langle 0.2,0\rangle \end{bmatrix}$, $D^2 Y D = D^2 = D Y D^2$ holds.

Hence D is 2-regular.

If $x.M = b$ is solvable,

let $x = \begin{bmatrix} \beta & \gamma \end{bmatrix}$, where $\beta = \begin{bmatrix} \langle 0.2,0.4\rangle & \langle 0.1,0.3\rangle \end{bmatrix}$ and $\gamma = \begin{bmatrix} \langle 0.2,0.4\rangle & \langle 0.2,0.5\rangle \end{bmatrix}$.

Thus $\beta A + \gamma C = c$ and $\beta B + \gamma D = d$.

Since $C = CA^{-}A$ and $B = BD^{-}D$, we get the equations,
$(\beta + \gamma CA^{-})A = C$ and $(\beta BD^{-} + \gamma)D = d$.

Now, $\beta + \gamma CA^{-} = \begin{bmatrix} \langle 0.2,0.4\rangle & \langle 0.1,0.3\rangle \end{bmatrix}$ and
$(\beta + \gamma CA^{-})A = \begin{bmatrix} \langle 0.2,0.3\rangle & \langle 0.1,0.3\rangle \end{bmatrix}$.
$(\beta + \gamma CA^{-})A = c = \begin{bmatrix} \langle 0.2,0.3\rangle & \langle 0.1,0.3\rangle \end{bmatrix}$.

Hence $y = \beta + \gamma CA^{-} = \begin{bmatrix} \langle 0.2,0.3\rangle & \langle 0.1,0.3\rangle \end{bmatrix}$ is a solution of $y.A = c$.

Now, $\beta BD^{-} + \gamma = \begin{bmatrix} \langle 0.2,0.3\rangle & \langle 0.2,0.3\rangle \end{bmatrix}$ and
$(\beta BD^{-} + \gamma)D = \begin{bmatrix} \langle 0.2,0.3\rangle & \langle 0.2,0.3\rangle \end{bmatrix}$
$(\beta BD^{-} + \gamma)D = d = \begin{bmatrix} \langle 0.2,0.3\rangle & \langle 0.2,0.3\rangle \end{bmatrix}$.

Hence $z = \beta BD^{-} + \gamma = \begin{bmatrix} \langle 0.2,0.3\rangle & \langle 0.2,0.3\rangle \end{bmatrix}$ is a solution of $z.D = d$.

Also $c = \begin{bmatrix} \langle 0.2,0.3\rangle & \langle 0.1,0.3\rangle \end{bmatrix}$,

$$dD^{-}C = \begin{bmatrix} \langle 0.2,0.3\rangle & \langle 0.2,0.3\rangle \end{bmatrix} \begin{bmatrix} \langle 0.2,0.3\rangle & \langle 0.1,0\rangle \\ \langle 0,0\rangle & \langle 0.2,0\rangle \end{bmatrix} \begin{bmatrix} \langle 0.2,0.2\rangle & \langle 0.1,0.2\rangle \\ \langle 0,0.2\rangle & \langle 0,0.2\rangle \end{bmatrix}$$

$dD^{-}C = \begin{bmatrix} \langle 0.2,0.3\rangle & \langle 0.1,0.3\rangle \end{bmatrix}$, and $d = \begin{bmatrix} \langle 0.2,0.3\rangle & \langle 0.2,0.3\rangle \end{bmatrix}$

$$cA^{-}B = \begin{bmatrix} \langle 0.2,0.3\rangle & \langle 0.1,0.3\rangle \end{bmatrix} \begin{bmatrix} \langle 1,0\rangle & \langle 0,1\rangle \\ \langle 0,0\rangle & \langle 0.2,0\rangle \end{bmatrix} \begin{bmatrix} \langle 0.2,0.4\rangle & \langle 0.1,0.4\rangle \\ \langle 0.2,0.3\rangle & \langle 0.2,0.3\rangle \end{bmatrix}$$

$cA^{-}B = \begin{bmatrix} \langle 0.2,0.3\rangle & \langle 0.1,0.3\rangle \end{bmatrix}$.

We know that $A \leq B \Leftrightarrow a_{ij\mu} \leq b_{ij\mu}$ and $a_{ij\vartheta} \geq b_{ij\vartheta}$, for all i and j.

From the above definition, $dD^{-}C \leq c$ and $cA^{-}B \leq d$.

Theorem 4.4.

Let $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ with A and D are right k - regular IFMs. If $yA^{k} = c$ and $zD^{k} = d$ are solvable, $c \geq dD^{-}C^{k}$ and $d \geq cA^{-}B^{k}$ then $xM^{k} = b$ is solvable where $M^{k} = \begin{bmatrix} A^{k} & B^{k} \\ C^{k} & D^{k} \end{bmatrix}$, $x = \begin{pmatrix} y & z \end{pmatrix}$ and $b = \begin{pmatrix} c & d \end{pmatrix}$.

Proof:

Since $yA^{k} = c$ and $zD^{k} = d$ are solvable, let $y = cA^{-}$ and $z = dD^{-}$ are the solutions $\Rightarrow cA^{-}A^{k} = c$ and $dD^{-}D^{k} = d$.

From the given conditions, $c \geq dD^{-}C^{k}$ and $d \geq cA^{-}B^{k}$ we get, $c = c + dD^{-}C^{k}$ and $d = d + cA^{-}B^{k}$.

Now,

$$\begin{pmatrix} cA^{-} & dD^{-} \end{pmatrix} \begin{bmatrix} A^{k} & B^{k} \\ C^{k} & D^{k} \end{bmatrix} = \begin{pmatrix} cA^{-}A^{k} + dD^{-}C^{k} & CA^{-}B^{k} + dD^{-}D^{k} \end{pmatrix} = \begin{pmatrix} c + dD^{-}C^{k} & CA^{-}B^{k} + d \end{pmatrix} = \begin{pmatrix} c & d \end{pmatrix} = b$$

Thus $xM^{k} = b$ is solvable.
Hence the theorem.

Remark 4.5.

For $k = 1$, the Theorem [4.2] and [4.4] reduces to the following:

Theorem 4.6. [15]

Let $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ with A and D are regular IFMs, M/A and M/D exists, $R(C) \subseteq R(A)$ and $R(B) \subseteq R(D)$. Then $xM = b$ is solvable if and only if $y.A = c$ and $z.D = d$ are solvable $c \geq dD^{-}C$ and $d \geq cA^{-}B$.

Theorem 4.7.

Let $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ with A and D are left k-regular IFMs, M/A and M/D are exist. $C(B) \subseteq C(A^{k})$ and $C(C) \subseteq C(D^{k})$. If $Mx = d$ is solvable then $Ay = b$ and $Dz = c$ are solvable, where $d = \begin{pmatrix} b \\ c \end{pmatrix}$, $c \geq CA^{-}b$ and $b \geq BD^{-}c$.

Proof:

This can be proved along the same lines as that of Theorem (4.2) and hence omitted.

Theorem 4.8.

Let $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ with A and D are left k-regular IFMs. If $A^{k}y = b$ and $D^{k}z = c$ are solvable, $c \geq C^{k}A^{-}b$ and $b \geq B^{k}D^{-}c$ then $M^{k}x = d$ is solvable where $M^{k} = \begin{bmatrix} A^{k} & B^{k} \\ C^{k} & D^{k} \end{bmatrix}$, $x = \begin{pmatrix} y \\ z \end{pmatrix}$. and $d = \begin{pmatrix} b \\ c \end{pmatrix}$.

Proof:

Since $A^{k}y = b$ and $D^{k}z = c$ are solvable, let $y = A^{-}b$ and $z = D^{-}c$ are the solution $\Rightarrow A^{k}A^{-}b = b; D^{k}D^{-}c = c$.

From the given conditions, $c \geq C^{k}A^{-}b$ and $b \geq B^{k}D^{-}c$ we get, $c = c + C^{k}A^{-}b$ and $b = b + B^{k}D^{-}c$.
Now,

$$\begin{bmatrix} A^{k} & B^{k} \\ C^{k} & D^{k} \end{bmatrix} \begin{pmatrix} A^{-}b \\ D^{-}c \end{pmatrix} = \begin{pmatrix} A^{k}A^{-}b + B^{k}D^{-}c \\ C^{k}A^{-}b + D^{k}D^{-}c \end{pmatrix} = \begin{pmatrix} b + B^{k}D^{-}c \\ C^{k}A^{-}b + c \end{pmatrix} = \begin{pmatrix} b \\ c \end{pmatrix} = d$$

Thus $M^{k}x = d$ is solvable.

Hence the Theorem.

Remark 4.9.

For $k = 1$, the Theorem [4.7] and [4.8] reduces to the following.

Theorem 4.10.

Let $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ with A and D are regular IFMs, M/A and M/D exists. $C(C) \subseteq C(D)$ and $C(B) \subseteq C(A)$. Then $Mx = d$ is solvable iff $Ay = b$ and $Dz = c$ are solvable, $c \geq CA^{-}b$ and $b \geq BD^{-}c$.

Remark 4.11.

In particular, for $B = 0$, Theorem [4.2] and Theorem [4.7] reduces to the following.

Corollary 4.12.

For the intuitionistic fuzzy matrix

$M = \begin{bmatrix} A & 0 \\ C & D \end{bmatrix}$ with A and D are k-regular such that

$(i) R(C) \subseteq R(A^{k})$. If $xM = b$ is solvable then $yA = c$ and $zD = d$ are solvable.

$(ii) C(C) \subseteq C(D^k)$. If $Mx = d$ is solvable then $Ay = b$ and $Dz = c$ are solvable.

5. CONCLUSION

In this paper, the solution of fuzzy relational equations are determined in the case of k – regular intuitionistic fuzzy matrices. Also we introduce the concept of k - regularity for block intuitionistic fuzzy matrices and in this case, the consistency of intuitionistic fuzzy relational equations are discussed.

COMPETING INTERESTS

Authors have declared that no competing interests exist.

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