British Journal of Mathematics & Computer Science
19(1): 1-12, 2016, Article no.BJMCS.28720
ISSN: 2231-0851

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A Theorem for Zeros of Maximal Monotone and Bounded Maps in Certain Banach Spaces

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Authors' contributions

This work was carried out in collaboration between all authors. All authors made significant contribution. All authors read and approved the final manuscript.

Article Information

DOI: 10.9734/BJMCS/2016/28720
Editor(s):
(1) Morteza Seddighin, Indiana University East Richmond, USA.
Reviewers:
(1) Sofiya Ostrovska, Atilim University, Turkey.
(2) Lexter R. Natividad, Central Luzon State University, Philippines.
(3) George Chailos, University of Nicosia, Cyprus.
(4) Angelo Favini, Universit di Bologna, Italy.
Complete Peer review History: http://www.sciencedomain.org/review-history/16543

Received: *1st August 2016*
Accepted: *6th September 2016*
Published: *13th October 2016*

Original Research Article

Abstract

Let X be a $p-$uniformly convex and $q-$uniformly smooth real Banach space with dual space X^*. Let $T_1 : X \to 2^{X^*}$ and $T_2 : X \to 2^{X^*}$ be bounded maximal monotone mappings. An iterative process is constructed and proved to converge strongly to a zero of sum of the two maps.

Keywords: *Monotone mapping; maximal monotone mappings; uniformly convex space; uniformly smooth space; strong convergence.*

2010 Mathematics Subject Classification: 47H04,47H05,47J25.

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1 Introduction

Let H be a real Hilbert space. A map $T : H \to 2^H$ is called *monotone* if for each $x, y \in H$, the following inequality holds:

$$\langle \xi - \tau, x - y\rangle \geq 0 \ \ \forall\ \xi \in Tx, \tau \in Ty. \tag{1.1}$$

This monotonicity condition in Hilbert space has been extended to arbitrary normed linear spaces. To introduce one of two known and studied extensions, we need the following definition. Let E be a real normed space with dual space E^*. A mapping $J : E \to 2^{E^*}$ is called the *normalised duality map* if for each $x \in E$,

$$Jx = \{x^* \in E^* : \langle x, x^*\rangle = ||x||||x^*||, ||x|| = ||x^*||\}.$$

A map $T : E \to 2^E$ is called *accretive* if for each $x, y \in E$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle \xi - \tau, j(x - y)\rangle \geq 0, \ \ \forall\ \xi \in Tx, \tau \in Ty. \tag{1.2}$$

It is well known that if E is a real Hilbert space, then $J = I$, the identity map on E. In this case, the inequality (1.2) reduces to inequality (1.1). Hence, accretivity in normed spaces is one extension of Hilbert space monotonicity condition to arbitrary real normed spaces.

The map T is called *maximal accretive* if it is accretive and, in addition, the graph of T is not properly contained in the graph of any other accretive operator. In other words, a map T is maximal accretive if and only if T is accretive and $R(I + tT) = E$ for all $t > 0$. If $E = H$, a real Hilbert space, *maximal accretive* mappings are called *maximal monotone*.

Also, a mapping $T : E \to 2^{E^*}$ is called *monotone* if for all $x, y \in D(T)$

$$\langle \xi - \zeta, x - y\rangle \geq 0 \ \ \forall \xi \in Tx, \ \forall \zeta \in Ty. \tag{1.3}$$

It is clear that if $E = H$ a real Hilbert space, then $E = E^* = H$ and inequality (1.3) coincides with the monotonicity definition in Hilbert spaces. So, this is another extension of Hilbert space monotonicity.

A mapping $T : X \to 2^{X^*}$ is said to be maximal monotone if it is monotone and for $(x, u) \in X \times X^*$ the inequalities $\langle u - v, x - y\rangle \geq 0$, for all $(y, v) \in G(T)$, imply $(x, u) \in G(T)$ where $G(T)$ is the graph of T.

A fundamental problem in the study of monotone operators in Hilbert spaces is that of finding an element $u \in H$ such that $0 \in Tu$. This problem has been investigated by many researchers.

The *proximal point algorithm (PPA)* introduced by Martinet [1] and studied extensively by Rockafellar [2] and numerous authors is concerned with an iterative method for approximating a solution of $0 \in Tu$ where T is a maximal monotone operator. Specifically, given $x_n \in H$, the proximal point algorithm generates the next iterate x_{n+1} by solving the following equation:

$$x_{n+1} = \Big(I + \frac{1}{\lambda_n}T\Big)^{-1}(x_n) + e_n, \tag{1.4}$$

where $\lambda_n > 0$ is a regularizing parameter.

Rockafellar [2] proved that if the sequence $\{\lambda_n\}_{n=1}^{\infty}$ is bounded from above, then the resulting sequence $\{x_n\}_{n=1}^{\infty}$ of proximal point iterates converges *weakly* to a solution of $0 \in Tu$, *provided that a solution exists.* He then posed the following question.

> **Question i.** Does the proximal point algorithm always converge strongly?

Güler [3], (see also Bauschke et al. [4]) resolved this question in the negative. This raised the following question naturally.

> **Question ii.** Can the proximal point algorithm be modified to guarantee strong convergence?

Authors like Bruck [5],Solodov and Svaiter [6], Kamimura and Takahashi [7], Reich and Sabach [8], **Xu** [9] and Lehdili and Moudafi [10] obtained modifications of the *PPA* that yield strong convergence.

Remark 1.1. Observe that in using the proximal point algorithm, *at each step of the iteration process*, one has to compute $\left(I + \frac{1}{\lambda_n}T\right)^{-1}(x_n)$ and this may not generally be convenient in several applications.

Consequently, while thinking of modifications of the proximal point algorithm that will guarantee strong convergence, the following question may be, perhaps, more important than **Question ii**.

> **Question iii.** Can an iteration process be developed which will not involve the computation of $\left(I + \frac{1}{\lambda_n}T\right)^{-1}(x_n)$ at each step of the iteration process and which will still guarantee *strong convergence* to a solution of $0 \in Tu$?

In response to *Question iii*, Chidume and Djitte [11] gave an affirmative answer when the space E involved is a *2-uniformly smooth* real *Banach* space. In fact, they proved the following theorem.

Theorem 1.1. *Let E be a $2-$uniformly smooth real Banach space, and let $T : E \to E$ be a bounded $m-$accretive map.For arbitrary $x_1 \in E$, define a sequence $\{x_n\}_{n=1}^{\infty}$ by,*

$$x_{n+1} = x_n - \lambda_n T x_n - \lambda_n \theta_n (x_n - x_1), \; n \geq 1,$$

where $\{\lambda_n\}_{n=1}^{\infty}$ and $\{\theta_n\}_{n=1}^{\infty}$ are sequences in $(0, 1)$ satisfying the folliowing conditions:

(i) $\lim_{n\to\infty} \theta_n = 0, \quad \{\theta_n\}_{n=1}^{\infty}$ *is decreasing;*
(ii) $\sum_{n=1}^{\infty} \lambda_n \theta_n = \infty, \quad \lambda_n = o(\theta_n)$;
(iii) $\lim_{n\to\infty} \frac{\left[\frac{\theta_{n-1}}{\theta_n} - 1\right]}{\lambda_n \theta_n} = 0, \quad \sum_{n=1}^{\infty} \lambda_n^2 < \infty$. *There exists a constant $\gamma_0 > 0$ such that if $\lambda_n \leq \gamma_0 \theta_n$ for all $n \geq 1$. Then the sequence $\{x_n\}_{n=1}^{\infty}$ converges strongly to a solution of the equation $Tx = 0$.*

Remark 1.2. We note here that L_p spaces, $p \geq 2$ are 2-uniformly smooth but L_p spaces, $p \in (1, 2)$ are not. So, this theorem of Chidume and Djitte does not guarantee strong convergence to a solution of the equation $Tu = 0$ on L_p spaces, for $p \in (1, 2)$.

Remark 1.3. A solution of $0 \in Tu$ where T is an accretive-type multi-valued mapping, in general, corresponds to an equilibrium state of a dynamical system (see e.g., Zeidler, [12]).

We now consider the inclusion $0 \in Tu$ where $T : E \to 2^{E^*}$ is of the monotone-type and E is a real normed space. Assuming $f : E \to \mathbb{R} \cup \{\infty\}$ is a proper convex function, then, the subdifferential of $f, \partial f : E \to 2^{E^*}$ is defined as follows: for $x \in E$,

$$\partial f(x) = \{x^* \in E^* : f(y) - f(x) \geq \langle y - x, x^* \rangle, \forall \; y \in E\}$$

and it is easy to see that $0 \in \partial f(u)$ if and only if u is a minimizer of f. It is equally easy to see that the subdifferential of $f, \partial f : E \to 2^{E^*}$ satitsfies the following inequality:

$$\langle x^* - y^*, x - y \rangle \geq 0 \ \forall x^* \in \partial f(x), y^* \in \partial f(y). \tag{1.5}$$

In this case, the subdifferential is said to be monotone.

Remark 1.4. If T is the subdifferential of a convex functional defined on the Banach space E, then a solution of $0 \in Tu$, corresponds to a minimizer of some convex functional defined on E.

Our objective in this paper is to prove a strong convergence theorem for sum of two *multi-valued maximal monotone and bounded mappings* in $p-$uniformly convex and $q-$uniformly smooth real Banach space. This complements theorem 1.1 and is applicable in all L_p spaces, $1 < p < \infty$. Our method of proof is different and of independent interest.

2 Preliminaries

Let E be a real normed linear space of dimension ≥ 2. The *modulus of smoothness* of E , $\rho_E : [0, \infty) \to [0, \infty)$, is defined by:

$$\rho_E(\tau) := \sup \left\{ \frac{\|x+y\| + \|x-y\|}{2} - 1 : \|x\| = 1, \|y\| = \tau, \quad \tau > 0 \right\}.$$

A normed linear space E is called *uniformly smooth* if

$$\lim_{\tau \to 0} \frac{\rho_E(\tau)}{\tau} = 0.$$

It is well known (see *e.g.*, Chidume [13] p. 16, also Lindenstrauss and Tzafriri [14]) that ρ_E is nondecreasing. If there exist a constant $c > 0$ and a real number $q > 1$ such that $\rho_E(\tau) \leq c\tau^q$, then E is said to be *q-uniformly smooth*. Typical examples of such spaces are the L_p, ℓ_p and W_p^m spaces for $1 < p < \infty$ where,

$$L_p \ (or \ l_p) \ or \ W_p^m \ \text{is} \ \begin{cases} 2 - \text{uniformly smooth} & \text{if} \quad 2 \leq p < \infty; \\ p - \text{uniformly smooth} & \text{if} \quad 1 < p < 2. \end{cases}$$

A Banach space E is said to be *strictly convex* if

$$\|x\| = \|y\| = 1, \quad x \neq y \implies \left\| \frac{x+y}{2} \right\| < 1.$$

The *modulus of convexity* of E is the function $\delta_E : (0, 2] \to [0, 1]$ defined by

$$\delta_E(\epsilon) := \inf \left\{ 1 - \left\| \frac{x+y}{2} \right\| : \|x\| = \|y\| = 1; \epsilon = \|x - y\| \right\}.$$

The space E is *uniformly convex* if and only if $\delta_E(\epsilon) > 0$ for every $\epsilon \in (0, 2]$.

It is also well known (see *e.g.*, Chidume [13] p. 34, also Lindenstrauss and Tzafriri [14]) that δ_E is nondecreasing. If there exist a constant $c > 0$ and a real number $p > 1$ such that $\delta_E(\epsilon) \geq c\epsilon^p$, then E is said to be *p-uniformly convex*. Typical examples of such spaces are the L_p, ℓ_p and W_p^m spaces for $1 < p < \infty$ where,

$$L_p \ (or \ l_p) \ or \ W_p^m \ \text{is} \ \begin{cases} p - \text{uniformly convex} & \text{if} \quad 2 \leq p < \infty; \\ 2 - \text{uniformly convex} & \text{if} \quad 1 < p < 2. \end{cases}$$

The norm of E is said to be *Fréchet differentiable* if for each $x \in S := \{u \in E : \|u\| = 1\}$,

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t},$$

exists and is attained uniformly for $y \in E$.

In what follows, we shall need the following definitions and results. Let E be a smooth real Banach space with dual E^*. The Lyapounov functional $\phi : E \times E \to \mathbb{R}$, is defined by,

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \text{ for } x, y \in E, \tag{2.1}$$

where J is the normalized duality mapping from E into E^*. It was introduced by Alber and has been studied by Alber [15], Alber and Guerre-Delabriere [16], Kamimura and Takahashi [7], Reich [17] and a host of other authors. If $E = H$, a real Hilbert space, then equation (2.1) reduces to $\phi(x, y) = \|x - y\|^2$ for $x, y \in H$. It is obvious from the definition of the function ϕ that

$$(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2 \text{ for } x, y \in E. \tag{2.2}$$

Define a map $V : X \times X^* \to \mathbb{R}$ by

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^*\rangle + \|x^*\|^2, \quad \text{for } x \in X,\ x^* \in X^*. \tag{2.3}$$

Then,

$$V(x, x^*) = \phi(x, J^{-1}(x^*))\ \forall x \in X,\ x^* \in X^*. \tag{2.4}$$

Lemma 2.1 (Alber, [18]). *Let X be a reflexive strictly convex and smooth Banach space with X^* as its dual. Then,*

$$V(x, x^*) + 2\langle J^{-1}x^* - x, y^*\rangle \leq V(x, x^* + y^*) \tag{2.5}$$

for all $x \in X$ and $x^*, y^* \in X^*$.

Lemma 2.2 (Alber, [18]). *Let X be a reflexive strictly convex and smooth Banach space with X^* as its dual. Let $W : X \times X \to \mathbb{R}^{\not k}$ be defined by $W(x, y) = \frac{1}{2}\phi(y, x)$. Then,*

$$W(x, y) - W(z, y) \geq \langle Jx - Jz, z - y\rangle, \tag{2.6}$$

i.e.,

$$\phi(y, x) - \phi(y, z) \geq 2\langle Jx - Jz, z - y\rangle, \tag{2.7}$$

and also

$$W(x, y) \leq \langle Jx - Jy, x - y\rangle, \tag{2.8}$$

for all $x, y, z \in X$.

Lemma 2.3 (Alber, [18]). *Let X be a uniformly convex Banach space. Then, for any $R > 0$ and any $x, y \in X$ such that $\|x\| \leq R$, $\|y\| \leq R$, the following inequality holds:*

$$\langle Jx - Jy, x - y\rangle \geq (2L)^{-1}\delta_X(c_2^{-1}\|x - y\|),$$

where $c_2 = 2max\{1, R\}$, $1 < L < 1.7$.

Lemma 2.4 (Alber, [18]). *Let X be a uniformly smooth and strictly convex Banach space. Then for any $R > 0$ and any $x, y \in X$ such that $\|x\| \leq R$, $\|y\| \leq R$ the following inequality holds:*

$$\langle Jx - Jy, x - y\rangle \geq (2L)^{-1}\delta_{X^*}(c_2^{-1}\|Jx - Jy\|),$$

where $c_2 = 2max\{1, R\}$, $1 < L < 1.7$.

Lemma 2.5 (Alber, [18]). *Let T_1 and T_2 be maximal monotone operators from X to 2^{X^*} and $D(T_1) \cap intD(T_2) \neq \emptyset$. Then their sum $T_1 + T_2$ is also a maximal monotone operator.*

Let E^* be a real strictly convex dual Banach space with a Fréchet differentiable norm. Let $T : E \to 2^{E^*}$ be a maximal monotone map with no monotone extension. Let $z \in E^*$ be fixed. Then, for every $\lambda > 0$ there exists a unique $x_\lambda \in E$ such that $Jx_\lambda + \lambda Tx_\lambda \ni z$ (see [19], p. 342). Setting $J_\lambda z = x_\lambda$, we have the *resolvent* $J_\lambda := (J + \lambda T)^{-1} : E^* \to E$ of T for every $\lambda > 0$. The following is a celebrated result of Reich.

Lemma 2.6 (Reich, [19])**.** *Let E^* be a strictly convex dual Banach space with a Fréchet differentiable norm, and let T be a maximal monotone map from E to E^* such that $T^{-1}0 \neq \emptyset$. Let $z \in E^*$ be arbitrary but fixed. For each $\lambda > 0$ there exists a unique $x_\lambda \in E$ such that $Jx_\lambda + \lambda Tx_\lambda \ni z$. Furthermore, x_λ converges strongly to a unique $p \in T^{-1}0$.*

Lemma 2.7. *From Lemma 2.6, setting $\lambda_n := \frac{1}{\theta_n}$ where $\theta_n \to 0$ as $n \to \infty$, $\theta_n \leq \theta_{n-1}$ $\forall$ $n \geq 1$, $z = Jv$ for some $v \in E$, and $y_n := \left(J + \frac{1}{\theta_n}T\right)^{-1} z$, we obtain that:*

$$Ty_n = \theta_n(Jv - Jy_n), \tag{2.9}$$

$$y_n \to y^* \in T^{-1}0,$$

where $T : E \to E^*$ is maximal monotone.

Remark 2.1. We observe that equation (2.9) yields

$$Jy_{n-1} - Jy_n + \frac{1}{\theta_n}\Big(Ty_{n-1} - Ty_n\Big) = \frac{\theta_{n-1} - \theta_n}{\theta_n}\Big(Jv - Jy_{n-1}\Big).$$

Taking the duality pairing of the LHS of this equation with $y_{n-1} - y_n$, and using the monotonicity of T we obtain that,

$$\langle Jy_{n-1} - Jy_n, y_{n-1} - y_n\rangle \quad \leq \quad \frac{\theta_{n-1} - \theta_n}{\theta_n}\Big\|Jv - Jy_{n-1}\Big\|\|y_{n-1} - y_n\|. \tag{2.10}$$

In a p-uniformly convex space, we have (see e.g., Chidume [13], p.34,) that, for some constant $r > 0$,

$$\delta_E(\epsilon) \geq r\epsilon^p \qquad \text{for } 0 < \epsilon \leq 2. \tag{2.11}$$

From lemma 2.3 and inequalities (2.10) and (2.11) we obtain that,

$$\|y_{n-1} - y_n\| \leq \left(\frac{\theta_{n-1} - \theta_n}{\theta_n}\right)^{1/p} C_1, \quad \text{ for some } C_1 > 0. \tag{2.12}$$

Similarly, from lemma 2.4 and inequalities (2.10) and (2.11) we obtain that

$$\|Jy_{n-1} - Jy_n\| \leq \left(\frac{\theta_{n-1} - \theta_n}{\theta_n}\right)^{1/p} C_2, \quad \text{for some } C_2 > 0. \tag{2.13}$$

Lemma 2.8 (Kamimura and Takahashi, [7])**.** *Let X be a real smooth and uniformly convex Banach space, and let $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ be two sequences of X. If either $\{x_n\}_{n=1}^\infty$ or $\{y_n\}_{n=1}^\infty$ is bounded and $\phi(x_n, y_n) \to 0$ as $n \to \infty$, then $\|x_n - y_n\| \to 0$ as $n \to \infty$.*

Lemma 2.9 (Xu, [20])**.** *Let $\{a_n\}_{n=0}^\infty$ be a sequence of non-negative real numbers satisfying the following condition*

$$a_{n+1} \leq (1 - \sigma_n)a_n + \sigma_n b_n + c_n,\ n \geq 0, \tag{2.14}$$

where $\{\sigma_n\}_{n=0}^\infty, \{b_n\}_{n=0}^\infty$ and $\{c_n\}_{n=0}^\infty$ satisfy the conditions:

(i) $\{\sigma_n\}_{n=0}^{\infty} \subset [0,1]$, $\sum_{n=1}^{\infty} \sigma_n = \infty$ *or equivalently,* $\prod_{n=1}^{\infty}(1-\sigma_n) = 0$;
(ii) $\limsup_{n\to\infty} b_n \leq 0$;
(iii) $c_n \geq 0$ $(n \geq 0)$, $\sum_{n=1}^{\infty} c_n < \infty$.
Then, $\lim_{n\to\infty} a_n = 0$.

Lemma 2.10. *Let E be a smooth real Banach space with dual E^* and the function $\phi : E \times E \to \mathbb{R}$ defined by,*

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad \textit{for } x,y \in E,$$

where J is the normalized duality mapping from E into 2^{E^*}. Then,

$$\phi(y,x) = \phi(x,y) + 2\langle x, Jy\rangle - 2\langle y, Jx\rangle. \tag{2.15}$$

3 Main Results

Theorem 3.1. *For $p > 1$, $q > 1$, let E be a p-uniformly convex and q-uniformly smooth real Banach space and let E^* be its dual. Let $T_1 : E \to 2^{E^*}$ and $T_2 : E \to 2^{E^*}$ be maximal monotone and bounded maps such that $(T_1+T_2)^{-1}(0) \neq \emptyset$ For arbitrary $u_1 \in E$, define a sequence $\{u_n\}$ iteratively by:*

$$u_{n+1} = J^{-1}(Ju_n - \alpha_n\xi_n - \alpha_n\tau_n - \alpha_n\theta_n(Ju_n - Ju_1)), \quad \xi_n \in T_1u_n, \quad \tau_n \in T_2u_n \ n \geq 1,$$

where $\{\lambda_n\}_{n=1}^{\infty}$ and $\{\theta_n\}_{n=1}^{\infty}$ are sequences in $(0,1)$ such that $\lim_{n\to\infty}\theta_n = 0$, $\lambda_n^{\frac{1}{p}} \leq \gamma_0\theta_n$ for all $n \geq 1$ and for some $\gamma_0 > 0$. Then, the sequence $\{u_n\}_{n=1}^{\infty}$ is bounded.

Proof. Since $(T_1+T_2)^{-1}(0) \neq \emptyset$, let $u^* \in (T_1+T_2)^{-1}(0)$. Then, there exists $r > 0$ such that

$$\frac{r}{5} \geq \phi(u^*, u_1).$$

Define $B := \{x \in E : \phi(x^*, x) \leq r\}$. We show that $\phi(u^*, u_n) \leq r$, $\forall\ n \geq 1$; we do this by induction. By construction, $\phi(u^*, u_1) \leq r$. Assume $\phi(u^*, u_n) \leq r$ for some $n \geq 1$.

We show that $\phi(u^*, u_{n+1}) \leq r$. Suppose, for contradiction, that this is not the case, i.e., suppose that $\phi(u^*, u_{n+1}) > r$. Since T_1 and T_2 are bounded, define

$$\begin{aligned}
M : &= \sup\{||\xi + \tau + \theta(Ju - Ju_1)|| : u \in B, \theta \in (0,1), \xi \in T_1u, \tau \in T_2u\} + 1,\\
K_1 : &= \sup\{||J^{-1}(Ju - \alpha\xi - \alpha\tau - \alpha\theta(Ju - Ju_1)) - u|| : u \in B, \alpha, \theta \in (0,1), \xi \in T_1u, \tau \in T_2u\} + 1,\\
K_2 : &= \sup\{||Ju - Ju_1|| : ||u|| \leq r_0\} + 1 \ for\ some\ r_0 > 0,\\
M^* &= \Big(\frac{2Lc_2^pMK_1}{r}\Big)^{\frac{1}{p}}\\
\gamma_0 : &= \min\Big\{1, \frac{r}{10M^*(M+K_2)}, \frac{r}{10MK_1}\Big\},
\end{aligned}$$

where c_2 and L are constants appearing in lemma 2.3.

From the recurrence relation,

$$||Ju_{n+1} - Ju_n|| = \lambda_n||\xi_n + \tau_n + \theta_n(Ju_n - Ju_1)|| \leq \lambda_nM. \tag{3.1}$$

Observe that using lemma 2.3, equations (2.11) and (3.1), we have

$$||u_{n+1} - u_n|| \leq \lambda_n^{\frac{1}{p}} M^*. \tag{3.2}$$

Take $y^* = \lambda_n \xi_n + \lambda_n \tau_n + \lambda_n \theta_n (Ju_n - Ju_1)$. Using lemma 2.1 and relation (2.4), we now compute as follows:

$$\begin{aligned}
& r < \phi(u^*, u_{n+1}) \\
= \quad & V(u^*, Ju_n - \lambda_n \xi_n - \lambda_n \tau_n - \lambda_n \theta_n (Ju_n - Ju_1)) \\
\leq \quad & V(u^*, Ju_n) - 2\langle u_{n+1} - u^*, \lambda_n \xi_n + \lambda_n \tau_n + \lambda_n \theta_n (Ju_n - Ju_1)\rangle \\
= \quad & V(u^*, Ju_n) - 2\lambda_n \langle u_{n+1} - u_n, \xi_n + \tau_n + \theta_n (Ju_n - Ju_1)\rangle \\
& -2\lambda_n \langle u_n - u^*, \xi_n + \tau_n + \theta_n (Ju_n - Ju_1)\rangle \\
\leq \quad & \phi(u^*, u_n) + 2\lambda_n \|u_{n+1} - u_n\| \|\xi_n + \tau_n + \theta_n (Ju_n - Ju_1)\| \\
& -2\lambda_n \langle u_n - u^*, \xi_n + \tau_n + \theta_n (Ju_n - Ju_1)\rangle.
\end{aligned}$$

Using inequality (3.2) and the fact that T_1 and T_2 are monotone, we obtain

$$\begin{aligned}
& \phi(u^*, u_{n+1}) \\
& \leq \phi(u^*, u_n) + 2\lambda_n (\lambda_n^{\frac{1}{p}}) M^* M - 2\lambda_n \theta_n \langle u_n - u^*, Ju_n - Ju_1\rangle.
\end{aligned} \tag{3.3}$$

But using the monotonicity of J, we have that

$$\begin{aligned}
& -2\lambda_n \theta_n \langle u_n - u^*, Ju_n - Ju_1\rangle \\
= \quad & -2\lambda_n \theta_n \langle u_n - u_{n+1}, Ju_n - Ju_{n+1}\rangle - 2\lambda_n \theta_n \langle u_n - u_{n+1}, Ju_{n+1} - Ju_1\rangle \\
& -2\lambda_n \theta_n \langle u_{n+1} - u^*, Ju_n - Ju_{n+1}\rangle - 2\lambda_n \theta_n \langle u_{n+1} - u^*, Ju_{n+1} - Ju_1\rangle \\
\leq \quad & 2\lambda_n \theta_n \|u_{n+1} - u_n\| K_2 + 2\lambda_n \theta_n \|Ju_{n+1} - Ju_n\| K_1 \\
& +2\lambda_n \theta_n \langle u_{n+1} - u^*, Ju_1 - Ju_{n+1}\rangle.
\end{aligned}$$

Using inequalities (3.2), (3.1) and inequality (2.7) in lemma 2.2 we have that

$$\begin{aligned}
& -2\lambda_n \theta_n \langle u_n - u^*, Ju_n - Ju_1\rangle \\
\leq \quad & 2\lambda_n \theta_n (\lambda_n^{\frac{1}{p}}) M^* K_2 + \lambda_n \theta_n \phi(u^*, u_1) - \lambda_n \theta_n \phi(u^*, u_{n+1}) + 2\lambda_n^2 \theta_n M K_1.
\end{aligned}$$

So, from inequality (3.3), we have, using conditions on θ_n and λ_n, that

$$\begin{aligned}
& r < \phi(u^*, u_{n+1}) \\
\leq \quad & \phi(u^*, u_n) - \lambda_n \theta_n \phi(u^*, u_{n+1}) + 2\lambda_n (\lambda_n^{\frac{1}{p}}) M^* (M + K_2) + \lambda_n \theta_n \phi(u^*, u_1) + 2\lambda_n^2 \theta_n M K_1 \\
\leq \quad & r - \lambda_n \theta_n r + \lambda_n \theta_n \frac{r}{5} + \lambda_n \theta_n \frac{r}{5} + \lambda_n \theta_n \frac{r}{5} < r.
\end{aligned}$$

This is a contradiction. Hence, $\phi(u^*, u_n) \leq r$ for all $n \geq 1$ and so the sequence $\{u_n\}_{n=1}^{\infty}$ is bounded. □

Theorem 3.2. *For $p > 1$, $q > 1$, let E be a p-uniformly convex and q-uniformly smooth real Banach space and let E^* be its dual. Let $T_1 : E \to 2^{E^*}$ and $T_2 : E \to 2^{E^*}$ be maximal monotone and bounded maps such that $(T_1 + T_2)^{-1}(0) \neq \emptyset$ and $D(T_1) \cap intD(T_2) \neq \emptyset$. For arbitrary $u_1 \in E$, define a sequence $\{u_n\}$ iteratively by:*

$$u_{n+1} = J^{-1}(Ju_n - \alpha_n \xi_n - \alpha_n \tau_n - \alpha_n \theta_n (Ju_n - Ju_1)), \quad \xi_n \in T_1 u_n, \quad \tau_n \in T_2 u_n \quad n \geq 1.$$

where $\{\lambda_n\}_{n=1}^{\infty}$ and $\{\theta_n\}_{n=1}^{\infty}$ are sequences in $(0, 1)$ such that $\sum_{n=1}^{\infty} \lambda_n \theta_n = \infty$, $\lim_{n\to\infty} \frac{\lambda_n^{\frac{1}{p}}}{\theta_n} = 0$ and $\lim_{n\to\infty} \frac{\left(\frac{\theta_{n-1} - \theta_n}{\theta_n}\right)^{\frac{1}{p}}}{\lambda_n \theta_n} = 0$. Then, the sequence $\{u_n\}_{n=1}^{\infty}$ converges strongly to a solution of $0 \in (T_1 + T_2)u$.

Proof. Using the same method of computation as in theorem 3.1, relation (2.15) and the fact that T_1 and T_2 are monotone we have, for some constant $M_1 > 0$,

$$\begin{aligned}
& \phi(y_n, u_{n+1}) \\
= \ & V(y_n, Ju_n - \lambda_n\xi_n - \lambda_n\tau_n - \lambda_n\theta_n(Ju_n - Ju_1)) \\
\leq \ & V(y_n, Ju_n) - 2\langle u_{n+1} - y_n, \lambda_n\xi_n + \lambda_n\tau_n + \lambda_n\theta_n(Ju_n - Ju_1)\rangle \\
= \ & \phi(u_n, y_n) + 2\langle u_n, Jy_n\rangle - 2\langle y_n, Ju_n\rangle - 2\langle u_{n+1} - y_n, \lambda_n\xi_n + \lambda_n\tau_n + \lambda_n\theta_n(Ju_n - Ju_1)\rangle \\
= \ & V(u_n, Jy_n) + 2\langle u_n, Jy_n\rangle - 2\langle y_n, Ju_n\rangle - 2\langle u_{n+1} - y_n, \lambda_n\xi_n + \lambda_n\tau_n + \lambda_n\theta_n(Ju_n - Ju_1)\rangle \\
\leq \ & V(u_n, Jy_{n-1}) - 2\langle y_n - u_n, Jy_{n-1} - Jy_n\rangle + 2\langle u_n, Jy_n\rangle - 2\langle y_n, Ju_n\rangle \\
& -2\langle u_{n+1} - y_n, \lambda_n\xi_n + \lambda_n\tau_n + \lambda_n\theta_n(Ju_n - Ju_1)\rangle \\
= \ & \phi(y_{n-1}, u_n) + 2\langle y_{n-1}, Ju_n\rangle - 2\langle u_n, Jy_{n-1}\rangle - 2\langle y_n - u_n, Jy_{n-1} - Jy_n\rangle \\
& +2\langle u_n, Jy_n\rangle - 2\langle y_n, Ju_n\rangle - 2\langle u_{n+1} - y_n, \lambda_n\xi_n + \lambda_n\tau_n + \lambda_n\theta_n(Ju_n - Ju_1)\rangle \\
= \ & \phi(y_{n-1}, u_n) + 2\langle y_{n-1} - y_n, Ju_n\rangle + 2\langle y_n, Jy_n - Jy_{n-1}\rangle \\
& -2\lambda_n\langle u_{n+1} - u_n, \xi_n + \tau_n + \theta_n(Ju_n - Ju_1)\rangle - 2\lambda_n\langle u_n - y_n, \xi_n + \tau_n + \theta_n(Ju_n - Ju_1)\rangle \\
\leq \ & \phi(y_{n-1}, u_n) + ||y_n - y_{n-1}||M_1 + ||Jy_n - Jy_{n-1}||M_1 + \lambda_n||u_{n+1} - u_n||M_1 \\
& -2\lambda_n\langle u_n - y_n, (\xi_n + \tau_n) - (\zeta_n + \mu_n)\rangle - 2\lambda_n\langle u_n - y_n, \zeta_n + \mu_n\rangle - 2\lambda_n\theta_n\langle u_n - y_n, Ju_n - Ju_1\rangle
\end{aligned}$$

for y_n as in lemma 2.7, $\zeta_n \in T_1y_n$ and $\mu_n \in T_2y_n$.

But for some constant $K^* > 0$, using equation (2.8), the fact that u_n and y_n are bounded,

$$\begin{aligned}
& -2\lambda_n\theta_n\langle u_n - y_n, Ju_n - Ju_1\rangle \\
= \ & -2\lambda_n\theta_n\langle u_n - y_{n-1}, Ju_n - Jy_{n-1}\rangle - 2\lambda_n\theta_n\langle y_{n-1} - y_n, Ju_n - Jy_{n-1}\rangle \\
& -2\lambda_n\theta_n\langle u_n - y_n, Jy_{n-1} - Ju_1\rangle \\
\leq \ & -\lambda_n\theta_n\phi(y_{n-1}, u_n) + \lambda_n\theta_n||y_n - y_{n-1}||K^* \\
& -2\lambda_n\theta_n\langle u_n - y_n, Jy_{n-1} - Jy_n\rangle - 2\lambda_n\theta_n\langle u_n - y_n, Jy_n - Ju_1\rangle \\
\leq \ & -\lambda_n\theta_n\phi(y_{n-1}, u_n) + \lambda_n\theta_n||y_n - y_{n-1}||K^* + \lambda_n\theta_n||Jy_{n-1} - Jy_n||K^* \\
& -2\lambda_n\theta_n\langle u_n - y_n, Jy_n - Ju_1\rangle.
\end{aligned}$$

Also, from lemma 2.5, $T_1 + T_2$ is maximal monotone and applying lemma 2.7, we have that

$$-2\lambda_n\langle u_n - y_n, \zeta_n + \mu_n\rangle - 2\lambda_n\theta_n\langle u_n - y_n, Jy_n - Ju_1\rangle = -2\lambda_n\langle u_n - y_n, (\zeta_n + \mu_n) + \theta_n(Jy_n - Ju_1)\rangle = 0.$$

Hence,

$$\begin{aligned}
& \phi(y_n, u_{n+1}) \\
\leq \ & \phi(y_{n-1}, u_n) - \lambda_n\theta_n\phi(y_{n-1}, u_n) + \lambda_n||u_{n+1} - u_n||M_1 + ||y_n - y_{n-1}||M_1 \\
& +||Jy_n - Jy_{n-1}||M_1 + \lambda_n\theta_n||Jy_n - Jy_{n-1}||K^* + \lambda_n\theta_n||y_n - y_{n-1}||K^*. \qquad (3.4)
\end{aligned}$$

Using inequalities (2.12), (2.13) and (3.2), together with conditions on θ_n and λ_n we have that,

$$\begin{aligned}
& \phi(y_n, u_{n+1}) \\
\leq \ & \phi(y_{n-1}, u_n) - \lambda_n\theta_n\phi(y_{n-1}, u_n) + \lambda_n(\lambda_n^{\frac{1}{p}})M^*M_1 + \Big(\frac{\theta_{n-1} - \theta_n}{\theta_n}\Big)^{\frac{1}{p}}C_1M_1 \\
+ \ & \Big(\frac{\theta_{n-1} - \theta_n}{\theta_n}\Big)^{\frac{1}{p}}C_2M_1 + \lambda_n\theta_n\Big(\frac{\theta_{n-1} - \theta_n}{\theta_n}\Big)^{\frac{1}{p}}C_2K^* + \lambda_n\theta_n\Big(\frac{\theta_{n-1} - \theta_n}{\theta_n}\Big)^{\frac{1}{p}}C_1K^* \\
\leq \ & \phi(y_{n-1}, u_n) - \lambda_n\theta_n\phi(y_{n-1}, u_n) + \lambda_n(\lambda_n^{\frac{1}{p}})K_0 + \Big(\frac{\theta_{n-1} - \theta_n}{\theta_n}\Big)^{\frac{1}{p}}K_0,
\end{aligned}$$

where $K_0 := max\{M^*M_1, C_1M_1 + C_2M_1 + C_1K^* + C_2K^*\}$.

Setting

$$a_n := \phi(y_{n-1}, u_n),\ \sigma_n := \lambda_n\theta_n,\ c_n \equiv 0,$$

and

$$b_n := K_0\left[\frac{\lambda_n^{\frac{1}{p}}}{\theta_n} + \frac{\left(\frac{\theta_{n-1}-\theta_n}{\theta_n}\right)^{\frac{1}{p}}}{\lambda_n\theta_n}\right]$$

we obtain that

$$a_{n+1} \leq (1-\sigma_n)a_n + \sigma_n b_n + c_n,\ n \geq 0.$$

It now follows from lemma 2.9 that $a_n \to 0$ as $n \to \infty, i.e., \phi(y_{n-1}, u_n) \to 0$ as $n \to \infty$. Consequently, by lemma 2.8, we have that $\lim||u_n - y_{n-1}|| = 0$. Since $y_n \to y^* \in (T_1+T_2)^{-1}(0)$, we have that u_n converges strongly to y^*. This completes the proof. □

Example 3.3. *Prototypes for our theorems are the following:*

$$\lambda_n = (n+1)^{-a} \text{ and } \theta_n = (n+1)^{-b},\quad n \geq 1,$$

where

$$0 < b < \frac{a}{r},\ a+b < \frac{1}{r},\ r = max\{p, q\}.$$

For example, without loss of generality, set $r = p$, *and take*

$$a := \frac{1}{(p+1)};\quad b := min\left\{\frac{1}{2}, \frac{1}{2p(p+1)}\right\},$$

Remark 3.1. (see e.g., Alber [18]) The analytical representations of duality mappings are known in a number of Banach spaces. For instance, in the spaces l^p, $L^p(G)$ and $W_m^p(G)$, $p \in (1, \infty)$, $p^{-1} + q^{-1} = 1$, respectively,

$$Jx = ||x||_{l^p}^{2-p} y \in l^q,\quad y = \{|x_1|^{p-2}x_1, |x_2|^{p-2}x_2, ...\},\quad x = \{x_1, x_2, ...\},$$

$$Jx = ||x||_{L^p}^{2-p}|x(s)|^{p-2}x(s) \in L^q(G),\quad s \in G,$$

and

$$Jx = ||x||_{W_m^p}^{2-p} \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha(|D^\alpha x(s)|^{p-2} D^\alpha x(s)) \in W_{-m}^q(G), m > 0, s \in G$$

.

4 Conclusion

Theorem 3.2 complements Theorem 1.1 to provide iterative process for the approximation of zeros of bounded maximal monotone operators. The result is applicable in all L_p spaces, $1 < p < \infty$.

Acknowledgement

The authors wish to thank the referees for their comments and suggestions.

Competing Interests

Authors have declared that no competing interests exist.

References

[1] Martinet B. Rgularization d' inéquations variationelles par approximations successive. Revue Francaise d'informatique et de Recherche Operationelle. 1970;4:154-159.

[2] Rockafellar RT. Monotone operator and the proximal point algorithm. SIAM J. Control Optim. 1976;14:877-898.

[3] Güler O. On the convergence of the proximal point algorithm for convex minimization. SIAM J. Control Optim. 1991;29:403-419.

[4] Bauschke Bruck HH, Matouskov E, Reich S. Projection and proximal point methods: Convergence results and counterexeamples. Nonlinear Anal. 2004;56:715-738J.

[5] Bruck Jr. RE. A strong convergent iterative solution of $0 \in U(x)$ for a maximal monotone operator U in Hilbert space. J. Math. Anal. Appl. 1974;48:114-126.

[6] Solodov MV, Svaiter BF. Forcing strong convergence of proximal point iterations in a Hilber space. Math. Program. Ser. 2000;A 87:189-202.

[7] Kamimura S, Takahashi W. Strong convergence of a proximal-type algorithm in a Banach space. SIAM J. Optimization. 2003;13(3):938-945.

[8] Reich S, Sabach S. Two strong convergence theorems for a proximal method in reflexive Banach spaces. Numerical Functional Analysis and Optimization. 2010;31:22-44.

[9] Xu HK. A regularization method for the proximal point algorithm. J. Global Opt. 2006;36:115-125.

[10] Lehdili N, Moudafi A. Combining the proximal algorithm and Tikhonov regularization. Optimization. 1996;37:239-252.

[11] Chidume CE, Djitte N. Strong convergence theorems for zeros of bounded maximal monotone nonlinear operators. Abstract and Applied Analysis. Volume 2012; Article ID 681348, 19 pages.
DOI: 10.1155/2012/681348.

[12] Zeidler E. Nonlinear functional analysis and its applications Part II: Monotone operators. Springer-Verlag, Berlin; 1985.

[13] Chidume CE. Geometric properties of banach spaces and nonlinear iterations. vol. 1965 of Lectures Notes in Mathematics, Springer, London, UK; 2009.

[14] Lindenstrauss J, Tzafriri L. Classical Banach spaces II: Function spaces. Ergebnisse Math. Grenzgebiete Bd. Springer-Verlag, Berlin. 1979;97.

[15] Ya. Alber. Metric and generalized projection operators in Banach spaces: Properties and applications. In Theory and Applications of Nonlinear Operators of Accretive and Monotone Type. (A. G. Kartsatos, Ed.), Marcel Dekker, New York. 1996;15-50.

[16] Ya. Alber, Guerre-Delabriere S. On the projection methods for fixed point problems. Analysis (Munich). 2001;21(1):17-39.

[17] Reich S. A weak convergence theorem for the alternating methods with Bergman distance. In: A. G. Kartsatos (Ed.), Theory and Applications of Nonlinear Operators of Accretive and Monotone Type, in Lecture notes in pure and Appl. Math. Dekker, New York. 1996;178:313-318.

[18] Ya. Alber, Ryazantseva I. Nonlinear Ill posed problems of monotone type Springer, London, UK; 2006.

[19] Reich S. Constructive techniques for accretive and monotone operators. Applied Non-Linear Analysis, Academic Press, New York. 1979;335-345.

[20] Xu HK. Iterative algorithms for nonlinear operators. J. London Math. Soc. 2002;66(2):240 - 256.

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