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The Convolution Sum $\sum_{m<n/8} \sigma_3(m)\sigma_3(n-8m)$

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Abstract

The convolution sum $\sum_{m<n/8} \sigma_3(m)\sigma_3(n-8m)$ is evaluated for all $n \in \mathbb{N}$. This evaluation is used to determine the formulae of the convolution sums $\sum_{m<\frac{n}{8}} \sigma_{3,0}(m;2)\sigma_{3,0}(n-8m;2)$, $\sum_{m<\frac{n}{8}} \sigma_3(m)\sigma_{3,1}(n-8m;2)$, etc., and the number of representations of n through the sum of eight triangular numbers.

Keywords: Divisor functions : Convolution sum

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1 Introduction

The study of arithmetical identities is classical in number theory and such investigations have been carried out by several mathematicians including the legend Srinivasa Ramanujan.

For $n \in \mathbb{N}$, $s, r \in \mathbb{N} \cup \{0\}$, $q \in \mathbb{C}$ with $|q| < 1$, we define some necessary divisor functions for later use, which also appear in many areas of number theory:

$$\sigma_s(n) = \sum_{d|n} d^s, \qquad \sigma_{s,r}(n;m) = \sum_{\substack{d|n \\ d \equiv r \bmod m}} d^s,$$

$$\Delta(q) := \sum_{n=1}^{\infty} \tau(n)q^n = q\prod_{n=1}^{\infty}(1-q^n)^{24}, \tag{1.1}$$

$$B(q) := \sum_{n=1}^{\infty} b(n)q^n = (\Delta(q)\Delta(q^2))^{\frac{1}{3}} = q\prod_{n=1}^{\infty}(1-q^n)^8(1-q^{2n})^8.$$

In general, it is satisfied that

$$b(n) = -8b(\frac{n}{2}) \tag{1.2}$$

for even n (see [1], Remark 4.3). We note that

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$$\sigma_s(n) = \sigma_{s,1}(n;2) + \sigma_{s,0}(n;2) \tag{1.3}$$

and

$$\sigma_{s,1}(2n;2) = \sigma_{s,1}(n;2), \quad \sigma_{s,0}(2n;2) = 2^s\sigma_s(n). \tag{1.4}$$

Let $q \in \mathbb{C}$ be such that $|q| < 1$. The Eisenstein series $L(q)$, $M(q)$, and $N(q)$ are

$$L(q) = 1 - 24\sum_{n=1}^{\infty}\sigma_1(n)q^n, \tag{1.5}$$

$$M(q) = 1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n, \tag{1.6}$$

$$N(q) = 1 - 504\sum_{n=1}^{\infty}\sigma_5(n)q^n, \tag{1.7}$$

see [2]. It was shown that

$$\Delta(q) = \frac{1}{1728}\left(M(q)^3 - N(q)^2\right) \tag{1.8}$$

by Ramanujan. And he gave in his notebook the following formulae, which are proved in [3]:

$$L(q) = (1-5x)w^2 + 12x(1-x)w\frac{dw}{dx}, \tag{1.9}$$

$$M(q) = (1+14x+x^2)w^4, \tag{1.10}$$

$$N(q) = (1+x)(1-34x+x^2)w^6, \tag{1.11}$$

where w is defined by

$$w = {}_2F_1(\frac{1}{2},\frac{1}{2};1;x) = \sum_{n=0}^{\infty}\frac{1}{2^{4n}}\binom{2n}{n}^2x^n.$$

From (1.8), (1.10), and (1.11), we obtain

$$\Delta(q) = \frac{x(1-x)^4w^{12}}{2^4}. \tag{1.12}$$

Applying the principle of duplication (see [3])

$$q \to q^2, \qquad x \to \left(\frac{1-\sqrt{1-x}}{1+\sqrt{1-x}}\right)^2, \qquad w \to \left(\frac{1+\sqrt{1-x}}{2}\right)w$$

to (1.10) and (1.12), we deduce that

$$M(q^2) = (1-x+x^2)w^4, \tag{1.13}$$

$$\Delta(q^2) = \frac{x^2(1-x)^2w^{12}}{2^8}. \tag{1.14}$$

Again applying the principle of duplication to (1.13) and (1.14), we obtain

$$M(q^4) = (1 - x + \frac{1}{16}x^2)w^4, \tag{1.15}$$

$$\Delta(q^4) = \frac{x^4(1-x)w^{12}}{2^{16}}. \tag{1.16}$$

Lastly, applying the principle of duplication to (1.15), we have

$$\begin{aligned} M(q^8) &= (\frac{17}{32} - \frac{17}{32}x + \frac{1}{256}x^2 + \frac{15}{32}\sqrt{1-x} - \frac{15}{64}x\sqrt{1-x})w^4 \\ &= -\frac{1}{32}M(q^2) + \frac{9}{16}M(q^4) + \frac{15}{32}\sqrt{1-x}w^4 - \frac{15}{64}x\sqrt{1-x}w^4. \end{aligned} \tag{1.17}$$

And from the above information K. S. Williams finds that

$$w^4 = \frac{1}{15}M(q) - \frac{2}{15}M(q^2) + \frac{16}{15}M(q^4), \tag{1.18}$$

$$xw^4 = \frac{1}{15}M(q) - \frac{1}{15}M(q^2), \tag{1.19}$$

$$x^2w^4 = \frac{16}{15}M(q^2) - \frac{16}{15}M(q^4), \tag{1.20}$$

in [4]. In this paper, we obtain the convolution sums

$$\sum_{m<\frac{n}{8}} \sigma_3(m)\sigma_3(n-8m), \quad \sum_{m<\frac{n}{8}} \sigma_{3,0}(m;2)\sigma_{3,0}(n-8m;2),$$
$$\sum_{m<\frac{n}{8}} \sigma_3(m)\sigma_{3,1}(n-8m;2), \quad \sum_{m<\frac{n}{8}} \sigma_{3,1}(m;2)\sigma_{3,0}(n-8m;2)$$

for $n \in \mathbb{N}$, using the Ramanujan and Williams' results. Furthermore, as an application of $\sum_{m<\frac{n}{8}} \sigma_3(m)\sigma_3(n-8m)$, we evaluate in Section 4 the number

$$\begin{aligned} \mathrm{T}(n) = \text{card}\Big\{(x_1, ..., x_{16}) \in \mathbb{N}_0^{16} \big| n &= \frac{1}{2}x_1(x_1+1) + \cdots + \frac{1}{2}x_8(x_8+1) \\ &+4\left(\frac{1}{2}x_9(x_9+1) + \cdots + \frac{1}{2}x_{16}(x_{16}+1)\right)\Big\}, \end{aligned}$$

for $n \in \mathbb{N}_0$ (see Theorem 4.1).

2 Preparations to Find $\sum_{m<\frac{n}{8}} \sigma_3(m)\sigma_3(n-8m)$

In order to calculate the convolution sum $\sum_{m<\frac{n}{8}} \sigma_3(m)\sigma_3(n-8m)$ we introduce the constants $g(n)$ and $h(n)$ for $n \in \mathbb{N}$ defined by

$$G(q) := \sum_{n=1}^{\infty} g(n)q^n = 2^4\left(\frac{\Delta(q^2)^{11}}{\Delta(q^4)^3\Delta(q)^4}\right)^{\frac{1}{6}} = x\sqrt{1-x}w^8 \tag{2.1}$$

and

$$H(q) := \sum_{n=1}^{\infty} h(n)q^n = 2^{12}\left(\frac{\Delta(q^4)^5}{\Delta(q^2)}\right)^{\frac{1}{6}} = x^3\sqrt{1-x}w^8. \tag{2.2}$$

Corollary 2.1. *We have*

(a)

$$G(q) = 2^4 q \prod_{n=1}^{\infty} \frac{(1+q^n)^{32}(1-q^n)^{16}}{(1+q^{2n})^{12}}.$$

(b)

$$H(q) = 2^{12} q^3 \prod_{n=1}^{\infty} (1+q^{2n})^4 (1-q^{4n})^{16}.$$

Proof. (a) By (1.1), (1.12), (1.14), (2.1), and (1.16), we obtain that

$$\begin{aligned}
G(q) = x\sqrt{1-x}w^8 &= 2^4 \left(\frac{\Delta(q^2)^{11}}{\Delta(q^4)^3 \Delta(q)^4}\right)^{\frac{1}{6}} \\
&= 2^4 \left(\frac{(q^2 \prod_{n=1}^{\infty}(1-q^{2n})^{24})^{11}}{(q^4 \prod_{n=1}^{\infty}(1-q^{4n})^{24})^3 (q \prod_{n=1}^{\infty}(1-q^n)^{24})^4}\right)^{\frac{1}{6}} \\
&= 2^4 q \prod_{n=1}^{\infty} \frac{(1-q^{2n})^{44}}{(1-q^{4n})^{12}(1-q^n)^{16}} = 2^4 q \prod_{n=1}^{\infty} \frac{(1-q^{2n})^{44}}{(1-q^{2n})^{12}(1+q^{2n})^{12}(1-q^n)^{16}} \\
&= 2^4 q \prod_{n=1}^{\infty} \frac{(1-q^{2n})^{32}}{(1+q^{2n})^{12}(1-q^n)^{16}} = 2^4 q \prod_{n=1}^{\infty} \frac{(1-q^n)^{32}(1+q^n)^{32}}{(1+q^{2n})^{12}(1-q^n)^{16}} \\
&= 2^4 q \prod_{n=1}^{\infty} \frac{(1+q^n)^{32}(1-q^n)^{16}}{(1+q^{2n})^{12}}.
\end{aligned}$$

(b) It is similar to Corollary 2.1 (a).

□

The first eighteen values of $b(n)$ are given in the Table 1

n	$b(n)$	n	$b(n)$	n	$b(n)$
1	1	7	1016	13	1382
2	−8	8	−512	14	−8128
3	12	9	−2043	15	−2520
4	64	10	1680	16	4096
5	−210	11	1092	17	14706
6	−96	12	768	18	16344

TABLE 1. $b(n)$ for n $(1 \le n \le 18)$

and $g(n)$ are in the Table 2

n	$g(n)$	n	$g(n)$	n	$g(n)$
1	16	7	−25728	13	233056
2	256	8	−49152	14	260096
3	1728	9	−44976	15	398976
4	6144	10	−53760	16	393216
5	10976	11	−55744	17	−301280
6	3072	12	73728	18	−523008

TABLE 2. $g(n)$ for n $(1 \leq n \leq 18)$

and $h(n)$ are given in the Table 3.

n	$h(n)$	n	$h(n)$	n	$h(n)$
1	0	7	-24576	13	540672
2	0	8	0	14	0
3	4096	9	-163840	15	385024
4	0	10	0	16	0
5	16384	11	-20480	17	-163840
6	0	12	0	18	0

TABLE 3. $h(n)$ for n $(1 \leq n \leq 18)$

We can see Proposition 2.1 in the wide area of [5].

Proposition 2.1. *We obtain*

(a) *(See [5])*

$$M^2(q) = 1 + 2^5 \cdot 3 \cdot 5 \sum_{n=1}^{\infty} \sigma_7(n) q^n.$$

(b) *(See [5], Theorm 5.1 (ii))*

$$M(q)M(q^2) = \frac{1}{17}M^2(q) + \frac{16}{17}M^2(q^2) + \frac{3600}{17}B(q).$$

(c) *(See [5], Theorm 5.1 (v))*

$$M(q)M(q^4) = \frac{1}{272}M^2(q) + \frac{15}{272}M^2(q^2) + \frac{16}{17}M^2(q^4) + \frac{4050}{17}B(q) + \frac{64800}{17}B(q^2).$$

By (1.1) and (1.12) we note that

$$B(q) = \frac{x(1-x)^2 w^8}{2^4}. \tag{2.3}$$

Lemma 2.2. *We have*

(a)

$$w^8 = \frac{1}{255}M^2(q) - \frac{2}{255}M^2(q^2) + \frac{256}{255}M^2(q^4) + \frac{512}{17}B(q) + \frac{8192}{17}B(q^2).$$

(b)

$$xw^8 = \frac{1}{255}M^2(q) - \frac{1}{255}M^2(q^2) + \frac{240}{17}B(q) + 256B(q^2).$$

(c)

$$x^2w^8 = \frac{1}{255}M^2(q) - \frac{1}{255}M^2(q^2) - \frac{32}{17}B(q).$$

(d)

$$x^3w^8 = \frac{1}{255}M^2(q) - \frac{1}{255}M^2(q^2) - \frac{32}{17}B(q) - 256B(q^2).$$

Proof. The proofs are similar, so we prove only Lemma 2.2 (a) and (b).

(a) From (1.18) we can calculate

$$\begin{aligned}
w^8 &= (w^4)^2 = \left(\frac{1}{15}M(q) - \frac{2}{15}M(q^2) + \frac{16}{15}M(q^4)\right)^2 \\
&= \left(\frac{1}{15}\right)^2 \{M^2(q) + 4M^2(q^2) + 256M^2(q^4) - 4M(q)M(q^2) \\
&\quad -64M(q^2)M(q^4) + 32M(q)M(q^4)\} \\
&= \left(\frac{1}{15}\right)^2 \Big[M^2(q) + 4M^2(q^2) + 256M^2(q^4) \\
&\quad - 4\left\{\frac{1}{17}M^2(q) + \frac{16}{17}M^2(q^2) + \frac{3600}{17}B(q)\right\} \\
&\quad - 64\left\{\frac{1}{17}M^2(q^2) + \frac{16}{17}M^2(q^4) + \frac{3600}{17}B(q^2)\right\} \\
&\quad +32\left\{\frac{1}{272}M^2(q) + \frac{15}{272}M^2(q^2) + \frac{16}{17}M^2(q^4) + \frac{4050}{17}B(q) + \frac{64800}{17}B(q^2)\right\}\Big],
\end{aligned}$$

where we insert Proposition 2.1 (b) and (c). So we obtain

$$w^8 = \frac{1}{255}M^2(q) - \frac{2}{255}M^2(q^2) + \frac{256}{255}M^2(q^4) + \frac{512}{17}B(q) + \frac{8192}{17}B(q^2).$$

(b) From (1.18) and (1.19), we can observe that

$$\begin{aligned}
xw^8 &= w^4 \cdot xw^4 \\
&= \left(\frac{1}{15}M(q) - \frac{2}{15}M(q^2) + \frac{16}{15}M(q^4)\right)\left(\frac{1}{15}M(q) - \frac{1}{15}M(q^2)\right) \\
&= \left(\frac{1}{15}\right)^2 \{M^2(q) - 3M(q)M(q^2) + 2M^2(q^2) + 16M(q)M(q^4) - 16M(q^2)M(q^4)\} \\
&= \left(\frac{1}{15}\right)^2 \Big[M^2(q) - 3\left\{\frac{1}{17}M^2(q) + \frac{16}{17}M^2(q^2) + \frac{3600}{17}B(q)\right\} + 2M^2(q^2) \\
&\quad + 16\left\{\frac{1}{272}M^2(q) + \frac{15}{272}M^2(q^2) + \frac{16}{17}M^2(q^4) + \frac{4050}{17}B(q) + \frac{64800}{17}B(q^2)\right\} \\
&\quad -16\left\{\frac{1}{17}M^2(q^2) + \frac{16}{17}M^2(q^4) + \frac{3600}{17}B(q^2)\right\}\Big] \\
&= \frac{1}{255}M^2(q) - \frac{1}{255}M^2(q^2) + \frac{240}{17}B(q) + 256B(q^2),
\end{aligned}$$

where we use Proposition 2.1 (b) and (c).

□

Lemma 2.3. *We have*

(a)

$$x^2\sqrt{1-x}w^8 = 8192B(q^4) + 256B(q^2) + \frac{1}{2}H(q).$$

(b)

$$\sqrt{1-x}w^8 = \frac{1}{255}M^2(q^2) - \frac{86}{85}M^2(q^4) + \frac{512}{255}M^2(q^8) - \frac{2208}{17}B(q^2) - \frac{70656}{17}B(q^4) + \frac{3}{2}G(q) - \frac{1}{4}H(q).$$

Proof. (a) First we apply the principle of duplication to $B(q)$ in (2.3) :

$$B(q^2) = \sum_{n=1}^{\infty} b(n)q^{2n} = \sum_{n=1}^{\infty} b(\frac{n}{2})q^n = \left(\Delta(q^2)\Delta(q^4)\right)^{1/3} = \frac{x^2(1-x)w^8}{2^8}. \tag{2.4}$$

Again applying the principle of duplication to (2.4), we have

$$\begin{aligned} B(q^4) &= \sum_{n=1}^{\infty} b(n)q^{4n} = \sum_{n=1}^{\infty} b(\frac{n}{4})q^n = \left(\Delta(q^4)\Delta(q^8)\right)^{1/3} \\ &= \frac{-2x^2w^8 + 2x^3w^8 + 2x^2\sqrt{1-x}w^8 - x^3\sqrt{1-x}w^8}{2^{14}}. \end{aligned} \tag{2.5}$$

This leads to

$$\begin{aligned} x^2\sqrt{1-x}w^8 &= 2^{13}B(q^4) + x^2w^8 - x^3w^8 + \frac{1}{2}x^3\sqrt{1-x}w^8 \\ &= 8192B(q^4) + 256B(q^2) + \frac{1}{2}H(q), \end{aligned}$$

where we use Lemma 2.2 (c), (d), and the definition of $H(q)$ in (2.2).

(b) By Lemma 2.2 (a) and (b), we can calculate $(1-x)w^8$ as follows :

$$(1-x)w^8 = w^8 - xw^8 = -\frac{1}{255}M^2(q^2) + \frac{256}{255}M^2(q^4) + 16B(q) + \frac{3840}{17}B(q^2). \tag{2.6}$$

Applying the principle of duplication to (2.6), we have

$$\begin{aligned} &\frac{1}{2}w^8 + \frac{1}{2}\sqrt{1-x}w^8 - xw^8 - \frac{3}{4}x\sqrt{1-x}w^8 + \frac{19}{32}x^2w^8 \\ &\quad + \frac{9}{32}x^2\sqrt{1-x}w^8 - \frac{3}{32}x^3w^8 - \frac{1}{64}x^3\sqrt{1-x}w^8 \\ &= -\frac{1}{255}M^2(q^4) + \frac{256}{255}M^2(q^8) + 16B(q^2) + \frac{3840}{17}B(q^4). \end{aligned} \tag{2.7}$$

Using Lemma 2.2 (a), (b), (c), (d) and Lemma 2.3 (a) into (2.7), we obtain

$$\begin{aligned} \frac{1}{2}\sqrt{1-x}w^8 - \frac{3}{4}x\sqrt{1-x}w^8 &= \frac{1}{510}M^2(q^2) - \frac{43}{85}M^2(q^4) + \frac{256}{255}M^2(q^8) \\ &\quad - \frac{1104}{17}B(q^2) - \frac{35328}{17}B(q^4) - \frac{1}{8}x^3\sqrt{1-x}w^8. \end{aligned}$$

Inserting the definitions of $G(q)$ and $H(q)$ into the above equation, we conclude that

$$\sqrt{1-x}w^8 = \frac{1}{255}M^2(q^2) - \frac{86}{85}M^2(q^4) + \frac{512}{255}M^2(q^8) - \frac{2208}{17}B(q^2) - \frac{70656}{17}B(q^4) + \frac{3}{2}G(q) - \frac{1}{4}H(q).$$

□

3 The Formula of $\sum_{m<\frac{n}{8}} \sigma_3(m)\sigma_3(n-8m)$

Theorem 3.1. *We have*

$$M(q)M(q^8) = \frac{1}{4352}M^2(q) + \frac{15}{4352}M^2(q^2) + \frac{15}{272}M^2(q^4) + \frac{16}{17}M^2(q^8) + \frac{17325}{136}B(q) + \frac{23175}{17}B(q^2) - \frac{424800}{17}B(q^4) + \frac{225}{32}G(q) - \frac{225}{128}H(q).$$

Proof. By (1.17), we observe that

$$\begin{aligned} M(q)M(q^8) &= M(q)\left\{-\frac{1}{32}M(q^2) + \frac{9}{16}M(q^4) + \frac{15}{32}\sqrt{1-x}w^4 - \frac{15}{64}x\sqrt{1-x}w^4\right\} \\ &= -\frac{1}{32}M(q)M(q^2) + \frac{9}{16}M(q)M(q^4) \\ &\quad + M(q)\left(\frac{15}{32}\sqrt{1-x}w^4 - \frac{15}{64}x\sqrt{1-x}w^4\right). \end{aligned} \tag{3.1}$$

Then, by (1.10), the third term of (3.1) can be written as

$$\begin{aligned} &M(q)\left(\frac{15}{32}\sqrt{1-x}w^4 - \frac{15}{64}x\sqrt{1-x}w^4\right) \\ &= (1+14x+x^2)w^4\left(\frac{15}{32}\sqrt{1-x}w^4 - \frac{15}{64}x\sqrt{1-x}w^4\right) \\ &= \frac{15}{32}\sqrt{1-x}w^8 + \frac{405}{64}x\sqrt{1-x}w^8 - \frac{45}{16}x^2\sqrt{1-x}w^8 - \frac{15}{64}x^3\sqrt{1-x}w^8 \\ &= \frac{1}{544}M^2(q^2) - \frac{129}{272}M^2(q^4) + \frac{16}{17}M^2(q^8) - \frac{13275}{17}B(q^2) - \frac{424800}{17}B(q^4) \\ &\quad + \frac{225}{32}G(q) - \frac{225}{128}H(q), \end{aligned}$$

where we use Lemma 2.3 (a), (b) and the definitions of $G(q)$ and $H(q)$. Therefore, by the help of Proposition 2.1 (b) and (c), Eq. (3.1) is

$$M(q)M(q^8) = \frac{1}{4352}M^2(q) + \frac{15}{4352}M^2(q^2) + \frac{15}{272}M^2(q^4) + \frac{16}{17}M^2(q^8) + \frac{17325}{136}B(q) + \frac{23175}{17}B(q^2) - \frac{424800}{17}B(q^4) + \frac{225}{32}G(q) - \frac{225}{128}H(q).$$

□

Theorem 3.2. *Let $n \in \mathbb{N}$. Then we have*

$$\begin{aligned} &\sum_{m<\frac{n}{8}} \sigma_3(m)\sigma_3(n-8m) \\ &= \frac{1}{8355840}\Big\{16\sigma_7(n) + 240\sigma_7(\frac{n}{2}) + 3840\sigma_7(\frac{n}{4}) + 65536\sigma_7(\frac{n}{8}) \\ &\quad - 34816\sigma_3(n) - 34816\sigma_3(\frac{n}{8}) + 18480b(n) + 197760b(\frac{n}{2}) \\ &\quad - 3624960b(\frac{n}{4}) + 1020g(n) - 255h(n)\Big\}. \end{aligned}$$

Proof. To prove we expand $M(q)M(q^8)$ in Theorem 3.1 with (1.6) :

$$\begin{aligned}
M(q)M(q^8) &= \left\{1 + 240\sum_{m=1}^{\infty}\sigma_3(m)q^m\right\}\left\{1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^{8n}\right\} \\
&= \left\{1 + 240\sum_{m=1}^{\infty}\sigma_3(m)q^m\right\}\left\{1 + 240\sum_{n=1}^{\infty}\sigma_3(\frac{n}{8})q^{n}\right\} \\
&= 1 + 240\sum_{m=1}^{\infty}\sigma_3(m)q^m + 240\sum_{n=1}^{\infty}\sigma_3(\frac{n}{8})q^n \\
&\quad + 240^2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sigma_3(m)\sigma_3(\frac{n}{8})q^{m+n}.
\end{aligned}$$

This leads to

$$\begin{aligned}
&\sum_{N=1}^{\infty}\left(\sum_{m<\frac{N}{8}}\sigma_3(m)\sigma_3(N-8m)\right)q^N \\
&= \frac{1}{240^2}\left\{M(q)M(q^8) - 1 - 240\sum_{N=1}^{\infty}\left(\sigma_3(N) + \sigma_3(\frac{N}{8})\right)q^N\right\},
\end{aligned}$$

since

$$\begin{aligned}
\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sigma_3(m)\sigma_3(\frac{n}{8})q^{m+n} &= \sum_{N=2}^{\infty}\sum_{m<N}\sigma_3(m)\sigma_3(\frac{N-m}{8})q^N \\
&= \sum_{N=1}^{\infty}\left(\sum_{m<\frac{N}{8}}\sigma_3(N-8m)\sigma_3(m)\right)q^N
\end{aligned}$$

by letting $m + n := N$. Finally, by using Proposition 2.1 (a), the proof is complete. □

Now we will find some convolution formulae based on Theorem 3.1 and Theorem 3.2.

Theorem 3.3. *Let* $n \in \mathbb{N}$*. Then we have*

(a)

$$\begin{aligned}
&\sum_{m<\frac{n}{8}}\sigma_{3,0}(m;2)\sigma_{3,0}(n-8m;2) \\
&= \frac{1}{130560}\Big\{16\sigma_7(\frac{n}{2}) + 240\sigma_7(\frac{n}{4}) + 3840\sigma_7(\frac{n}{8}) + 65536\sigma_7(\frac{n}{16}) \\
&\quad - 34816\sigma_3(\frac{n}{2}) - 34816\sigma_3(\frac{n}{16}) + 18480b(\frac{n}{2}) + 197760b(\frac{n}{4}) \\
&\quad -3624960b(\frac{n}{8}) + 1020g(\frac{n}{2}) - 255h(\frac{n}{2})\Big\}.
\end{aligned}$$

(b)

$$\sum_{m<\frac{n}{8}} \sigma_3(m)\sigma_{3,1}(n-8m;2)$$
$$= \frac{1}{8355840}\Big\{16\sigma_7(n) - 1808\sigma_7(\frac{n}{2}) - 26880\sigma_7(\frac{n}{4}) - 458752\sigma_7(\frac{n}{8})$$
$$- 34816\sigma_3(n) + 278528\sigma_3(\frac{n}{2}) + 243712\sigma_3(\frac{n}{8}) + 18480b(n)$$
$$-78720b(\frac{n}{2}) - 8048640b(\frac{n}{4}) + 1020g(n) - 255h(n)\Big\}.$$

(c)

$$\sum_{m<\frac{n}{8}} \sigma_{3,1}(m;2)\sigma_{3,0}(n-8m;2)$$
$$= \frac{1}{130560}\Big\{16\sigma_7(\frac{n}{2}) + 240\sigma_7(\frac{n}{4}) + 4352\sigma_7(\frac{n}{8}) - 65536\sigma_7(\frac{n}{16})$$
$$+ 30464\sigma_3(\frac{n}{2}) - 4352\sigma_3(\frac{n}{8}) + 34816\sigma_3(\frac{n}{16}) - 14160b(\frac{n}{2})$$
$$-128640b(\frac{n}{4}) + 3624960b(\frac{n}{8}) - 1020g(\frac{n}{2}) + 255h(\frac{n}{2})\Big\}.$$

Proof. (a) By (1.4), we note that

$$\sum_{m<\frac{n}{8}} \sigma_{3,0}(m;2)\sigma_{3,0}(n-8m;2) = \sum_{m<\frac{n}{8}} 2^3\sigma_3(\frac{m}{2})\cdot 2^3\sigma_3(\frac{n}{2}-4m)$$
$$= 2^6 \sum_{m<\frac{n}{16}} \sigma_3(m)\sigma_3(\frac{n}{2}-8m).$$

Thus we refer to Theorem 3.2.

(b) First we can expand (1.6) as

$$M(q) = 1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n = 1 + 240\sum_{n=1}^{\infty}\left(\sigma_{3,1}(n;2) + \sigma_{3,0}(n;2)\right)q^n$$
$$= 1 + 240\sum_{n=1}^{\infty}\sigma_{3,1}(n;2)q^n + 240\sum_{n=1}^{\infty}8\sigma_3(\frac{n}{2})q^n$$
$$= 1 + 240\sum_{n=1}^{\infty}\sigma_{3,1}(n;2)q^n + 8\cdot 240\sum_{n=1}^{\infty}\sigma_3(n)q^{2n}$$
$$= 8\left(1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^{2n}\right) - 7 + 240\sum_{n=1}^{\infty}\sigma_{3,1}(n;2)q^n$$
$$= 8M(q^2) - 7 + 240\sum_{n=1}^{\infty}\sigma_{3,1}(n;2)q^n.$$

This shows that

$$\sum_{n=1}^{\infty}\sigma_{3,1}(n;2)q^n = \frac{M(q) - 8M(q^2) + 7}{240}. \tag{3.2}$$

Second, by (1.6), we have

$$\sum_{n=1}^{\infty} \sigma_3(n) q^{8n} = \frac{M(q^8) - 1}{240}. \tag{3.3}$$

Multiplying (3.2) by (3.3) we obtain

$$\begin{aligned} &\left(\sum_{n=1}^{\infty} \sigma_{3,1}(n;2) q^n\right) \left(\sum_{m=1}^{\infty} \sigma_3(m) q^{8m}\right) \\ &= \frac{1}{240^2} \left(M(q) - 8M(q^2) + 7\right) \left(M(q^8) - 1\right) \\ &= \frac{1}{240^2} \left\{M(q)M(q^8) - 8M(q^2)M(q^8) + 7M(q^8) - M(q) + 8M(q^2) - 7\right\}. \end{aligned} \tag{3.4}$$

Then the left hand side of (3.4) is

$$\begin{aligned} \left(\sum_{n=1}^{\infty} \sigma_{3,1}(n;2) q^n\right) \left(\sum_{m=1}^{\infty} \sigma_3(m) q^{8m}\right) &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sigma_{3,1}(n;2) \sigma_3(m) q^{n+8m} \\ &= \sum_{N=1}^{\infty} \left(\sum_{m < \frac{N}{8}} \sigma_{3,1}(N - 8m; 2) \sigma_3(m) \right) q^N \end{aligned}$$

by letting $n + 8m := N$. And for the right hand side of (3.4) we use (1.6), Proposition 2.1 (a), (c), and Theorem 3.1.

(c) We note that

$$\begin{aligned} &\sum_{m < \frac{n}{8}} \sigma_3(m) \sigma_3(n - 8m) \\ &= \sum_{m < \frac{n}{8}} \{\sigma_{3,1}(m;2) + \sigma_{3,0}(m;2)\} \{\sigma_{3,1}(n - 8m;2) + \sigma_{3,0}(n - 8m;2)\} \\ &= \sum_{m < \frac{n}{8}} \sigma_{3,1}(m;2) \sigma_{3,1}(n - 8m;2) + \sum_{m < \frac{n}{8}} \sigma_{3,0}(m;2) \sigma_{3,1}(n - 8m;2) \\ &\quad + \sum_{m < \frac{n}{8}} \sigma_{3,1}(m;2) \sigma_{3,0}(n - 8m;2) + \sum_{m < \frac{n}{8}} \sigma_{3,0}(m;2) \sigma_{3,0}(n - 8m;2) \\ &= \sum_{m < \frac{n}{8}} \sigma_3(m) \sigma_{3,1}(n - 8m;2) + \sum_{m < \frac{n}{8}} \sigma_{3,1}(m;2) \sigma_{3,0}(n - 8m;2) \\ &\quad + \sum_{m < \frac{n}{8}} \sigma_{3,0}(m;2) \sigma_{3,0}(n - 8m;2). \end{aligned}$$

Thus we refer to Theorem 3.2, Theorem 3.3 (a), and (b).

□

4 The Number of Representations of n Through the Sum of Eight Triangular Numbers

The triangular numbers are the nonnegative integers

$$T_k = \frac{1}{2}k(k+1), \quad k \in \mathbb{N}_0,$$

so that

$$T_0 = 0, \quad T_1 = 1, \quad T_2 = 3, \quad T_3 = 6, \quad T_4 = 10, \quad T_5 = 15, ...$$

Let k be a positive integer. We denote the number of representations of n $(\in \mathbb{N}_0)$ as a sum of k triangular numbers by $t_k(n)$, that is,

$$t_k(n) := \text{card}\left\{(m_1, ..., m_k) \in \mathbb{N}_0^k \middle| n = \frac{1}{2}m_1(m_1+1) + \cdots + \frac{1}{2}m_k(m_k+1)\right\}.$$

Obviously, $t_k(0) = 1$ for all $k \in \mathbb{N}$.

Proposition 4.1. *Let $n \in \mathbb{N}_0$. Then we have*

(a) *(See [6], Theorem 16.1)*

$$t_2(n) = \sum_{\substack{d \in \mathbb{N} \\ d|(4n+1)}} \left(\frac{-4}{d}\right).$$

(b) *(See [6], Theorem 16.7)*

$$t_4(n) = \sigma_1(2n+1).$$

(c) *(See [6], Theorem 16.12)*

$$t_6(n) = -\frac{1}{8} \sum_{\substack{d \in \mathbb{N} \\ d|(4n+3)}} \left(\frac{-4}{d}\right) d^2.$$

(d) *(See [6], Theorem 16.13)*

$$t_8(n) = \sigma_3(n+1) - \sigma_3(\frac{n+1}{2}).$$

Theorem 4.1. *Let $n \in \mathbb{N}_0$. Then we have*

$$\mathrm{T}(n) = \frac{1}{557056}\Big\{16\sigma_7(n+5) - 16\sigma_7(\frac{n+5}{2}) + 1072b(n+5) + 23936b(\frac{n+5}{2}) \\ +278528b(\frac{n+5}{4}) - 68g(n+5) + 17h(n+5)\Big\}.$$

Proof. First we note that

$$\mathrm{T}(n) = \sum_{\substack{(l,m)\in\mathbb{N}_0^2 \\ l+4m=n}} t_8(l)t_8(m) = \sum_{0\leq m\leq \frac{n}{4}} t_8(n-4m)t_8(m).$$

By Proposition 4.1 (d), the above equation can be written as

$$\begin{aligned}
&\sum_{0\leq m\leq \frac{n}{4}} t_8(m)t_8(n-4m) \\
&= \sum_{0\leq m\leq \frac{n}{4}} \left\{\sigma_3(m+1)-\sigma_3(\frac{m+1}{2})\right\}\left\{\sigma_3(n-4m+1)-\sigma_3(\frac{n-4m+1}{2})\right\} \\
&= \sum_{0\leq m\leq \frac{n}{4}} \sigma_3(m+1)\sigma_3(n-4m+1) - \sum_{0\leq m\leq \frac{n}{4}} \sigma_3(m+1)\sigma_3(\frac{n+1}{2}-2m) \\
&\quad - \sum_{0\leq m\leq \frac{n}{4}} \sigma_3(\frac{m+1}{2})\sigma_3(n-4m+1) + \sum_{0\leq m\leq \frac{n}{4}} \sigma_3(\frac{m+1}{2})\sigma_3(\frac{n+1}{2}-2m) \\
&= \sum_{1\leq m\leq \frac{n}{4}+1} \sigma_3(m)\sigma_3(n-4(m-1)+1) - \sum_{1\leq m\leq \frac{n}{4}+1} \sigma_3(m)\sigma_3(\frac{n+1}{2}-2(m-1)) \\
&\quad - \sum_{1\leq m\leq \frac{n}{4}+1} \sigma_3(\frac{m}{2})\sigma_3(n-4(m-1)+1) \\
&\quad + \sum_{1\leq m\leq \frac{n}{4}+1} \sigma_3(\frac{m}{2})\sigma_3(\frac{n+1}{2}-2(m-1)) \\
&= \sum_{1\leq m\leq \frac{n}{4}+1} \sigma_3(m)\sigma_3(n+5-4m) - \sum_{1\leq m\leq \frac{n+3}{4}} \sigma_3(m)\sigma_3(\frac{n+5}{2}-2m) \\
&\quad - \sum_{1\leq m\leq \frac{n+4}{8}} \sigma_3(m)\sigma_3(n+5-8m) + \sum_{1\leq m\leq \frac{n+3}{8}} \sigma_3(m)\sigma_3(\frac{n+5}{2}-4m).
\end{aligned}$$

Therefore, we refer to

$$\begin{aligned}
&\sum_{m<\frac{n}{2}} \sigma_3(m)\sigma_3(n-2m) \\
&\quad = \frac{1}{2040}\sigma_7(n) + \frac{2}{255}\sigma_7(\frac{n}{2}) - \frac{1}{240}\sigma_3(n) - \frac{1}{240}\sigma_3(\frac{n}{2}) + \frac{1}{272}b(n),
\end{aligned}$$

in ([5], Theorem 5.2 (ii)),

$$\begin{aligned}
\sum_{m<\frac{n}{4}} \sigma_3(m)\sigma_3(n-4m) &= \frac{1}{32640}\sigma_7(n) + \frac{1}{2176}\sigma_7(\frac{n}{2}) + \frac{2}{255}\sigma_7(\frac{n}{4}) \\
&\quad - \frac{1}{240}\sigma_3(n) - \frac{1}{240}\sigma_3(\frac{n}{4}) + \frac{9}{2176}b(n) + \frac{9}{136}b(\frac{n}{2}),
\end{aligned}$$

in ([5], Theorem 5.2 (v)), and Theorem 3.2.

□

Example 4.2. *The values of* $\mathrm{T}(n)$ *for* $n = 1, 2, ..., 10$ *are as follows :*

n	1	2	3	4	5	6	7	8	9	10
$\mathrm{T}(n)$	8	28	64	134	288	568	1024	1793	3024	4868

TABLE 4. $\mathrm{T}(n)$ *for* n *(*$1 \le n \le 10$*)*

Theorem 4.3. *Let* $n \in \mathbb{N}$. *Then we have*

$$b(n) \equiv 2\sigma_7(2n) - 2\sigma_7(n) \pmod{17}.$$

In particular, if n *is odd, then*

$$b(n) \equiv \sigma_7(n) \pmod{17}.$$

Proof. First for $n = 1, 2$, it is obvious that $b(1) \equiv 2\sigma_7(2) - 2\sigma_7(1) \pmod{17}$ and $b(2) \equiv 2\sigma_7(4) - 2\sigma_7(2) \pmod{17}$ by the Table 1.

Second for $n \ge 3$, let us consider Theorem 4.1 as follows :

$$\begin{aligned} 557056\mathrm{T}(n) = 16\sigma_7(n+5) - 16\sigma_7(\frac{n+5}{2}) + 1072b(n+5) + 23936b(\frac{n+5}{2}) \\ + 278528b(\frac{n+5}{4}) - 68g(n+5) + 17h(n+5). \end{aligned} \tag{4.1}$$

For $N \in \mathbb{N}$, we can put $n = 2N - 1$ in (4.1) so we obtain

$$\begin{aligned} 2^{15} \cdot 17\mathrm{T}(2N-1) = {} & 16\sigma_7(2N+4) - 16\sigma_7(N+2) + 1072b(2N+4) \\ & + 2^7 \cdot 11 \cdot 17b(N+2) + 2^{14} \cdot 17b(\frac{N}{2}+1) - 2^2 \cdot 17g(2N+4) \\ & + 17h(2N+4). \end{aligned}$$

This leads us that

$$\begin{aligned} 0 & \equiv -\sigma_7(2N+4) + \sigma_7(N+2) + b(2N+4) \\ & \equiv -\sigma_7(2(N+2)) + \sigma_7(N+2) - 8b(N+2) \pmod{17}, \end{aligned}$$

where we use (1.2). Therefore we have

$$8b(N+2) \equiv -\sigma_7(2(N+2)) + \sigma_7(N+2) \pmod{17}. \tag{4.2}$$

Since $(15, 17) = 1$, (4.2) can be written as

$$\begin{aligned} 15 \cdot 8b(N+2) & \equiv -15\sigma_7(2(N+2)) + 15\sigma_7(N+2) \\ & \equiv 2\sigma_7(2(N+2)) - 2\sigma_7(N+2) \pmod{17} \end{aligned}$$

and so

$$b(N+2) \equiv 2\sigma_7(2(N+2)) - 2\sigma_7(N+2) \pmod{17} \quad \text{for} \quad N \in \mathbb{N}.$$

Thus we conclude that

$$b(n) \equiv 2\sigma_7(2n) - 2\sigma_7(n) \pmod{17} \quad \text{for} \quad n \in \mathbb{N}. \tag{4.3}$$

In particular, if n is odd, then (4.3) becomes

$$b(n) \equiv 2(1+2^7)\sigma_7(n) - 2\sigma_7(n) \equiv \sigma_7(n) \pmod{17},$$

and this special case is also seen in ([1], Lemma 4.6).

□

5 Conclusions

In this paper, we evaluate the convolution sum $\sum_{m<n/8} \sigma_3(m)\sigma_3(n-8m)$ for all $n \in \mathbb{N}$, which is available us to find the formulae of the convolution sums

$$\sum_{m<\frac{n}{8}} \sigma_{3,0}(m;2)\sigma_{3,0}(n-8m;2), \quad \sum_{m<\frac{n}{8}} \sigma_3(m)\sigma_{3,1}(n-8m;2),$$

etc., and the number of representations of n through the sum of eight triangular numbers.

Competing Interests

The author declares that no competing interests exist.

References

[1] Kim A, Kim D, Ikikardes NY. Remarks of congruent arithmetic sums of theta functions derived from divisor functions. Honam Mathematical J. 2013;35(3):351-372.

[2] Berndt BC. Ramanujan's Notebooks. Part II. Springer-Verlag, New York; 1989.

[3] Berndt BC. Ramanujan's Notebooks. Part III. Springer-Verlag, New York. 1991.

[4] Williams KS. The convolution sum $\sum_{m<n/8} \sigma(m)\sigma(n-8m)$. Pacific Journal of Mathematics. 2006;228(2):387-396.

[5] Cheng N, Williams KS. Evaluation of some convolution sums involving the sum of divisors functions. Yokohama Mathematical J. 2005;52:39-57.

[6] Williams KS. Number Theory in the Spirit of Liouville. London Mathematical Society, Student Texts 76, Cambridge; 2011.

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