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Products and Eccentric Diagraphs

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Abstract

The eccentricity $e(u)$ of a vertex u is the maximum distance of u to any other vertex of G. A vertex v is an eccentric vertex of vertex u if the distance from u to v is equal to $e(u)$. The eccentric digraph $ED(G)$ of a graph(digraph) G is the digraph that has the same vertex as G and an arc from u to v exists in $ED(G)$ if and only if v is an eccentric vertex of u in G. In this paper, we consider the eccentric digraphs of different products of graphs, viz., cartesian, normal, lexicographic, prism, etc.

Keywords: Eccentric vertex, eccentric digraph, cartesian product, lexicographic product, normal product, prism of graphs.

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1 Introduction

Many classes of graphs like hypercubes, Hamming graphs, prisms, etc. which find enormous applications in wide variety of fields like natural and social sciences, computer science, engineering, etc. are graph products themselves or are closely related to them. The development of large-scale networks has been graph theoretically supported by products. Different kinds of products act as powerful tools to construct bigger(wider) graphs, given smaller ordered/sized graphs. Many parameters are tested for the products in literature [1], [2], [3], etc. Cartesian product has been widely used by graph theorists and others too. Recently, a monograph by Imrich et al. [4] on graphs and their cartesian products reiterate the importance of the concept. There are other products like, normal product, lexicographic product, prisms, etc., which have also found lot of applications in many fields.

Distance in graphs is another major area for applications, which is an under current for various concepts. Many distance-based topological invariants like Wiener index, Siezed index, eccentric connectivity index, etc. find applications in Mathematical Chemistry. Many binary relations are defined by distances in a graph, viz., antipodal digraphs [5], eccentric graphs [6], eccentric digraphs [7], etc. These relations can be represented as smaller sized graphs/digraphs(compared to the original ones) so that they are easy to handle. Many properties of original graph are retained in

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these, but not all. So, the study of these graphs is interesting. One such concept, the eccentric digraphs is considered in this article for the product of graphs. The concept of eccentric digraphs of graphs was introduced more than a decade ago by Buckley [7]. It was generalized for digraphs by Boland and Miller [8]. We can find many articles on these in recent years by Gimbert et al. [9], [10], Boland et al. [11], Medha Itagi Huilgol et al. [12], [13], [14].

In this article we shall consider both directed graphs and symmetric digraphs. Unless mentioned otherwise for terminology and notation, the reader may refer Buckley and Harary [15] and Chartrand and Lesnaik [16], new ones will be introduced as and when required.

A directed graph or digraph G consists of a finite nonempty set $V(G)$ called *vertex set* with vertices and *edge set* $E(G)$ of ordered pairs of vertices called *arcs*; that is, $E(G)$ represents a binary relation on $V(G)$. A graph is a *symmetric digraph*, if in G for any arc $(u, v) \in E(G)$ implies $(v, u) \in E(G)$. If (u, v) is an arc, it is said that *u* is adjacent to *v* and also that *v* is adjacent from *u*. The set of vertices which are adjacent from (to) a given vertex *v* is denoted by $N^+(u)[N^-(u)]$ and its cardinality is the *out-degree* of *v*[*in-degree* of v]. A *walk* of length *k* from a vertex *u* to a vertex *v* in G is a sequence of vertices $u = u_0, u_1, u_2, \ldots, u_{k-1}, u_k = v$ such that each pair (u_{i-1}, u_i) is an *arc* of *G*. A digraph *G* is *strongly connected* if there is a *u* to *v* walk for any pair of vertices *u* and *v* of *G*. The *distance d(u,v)* from *u* to *v* is the length of a shortest *u* to *v* walk. The *eccentricity e(v)* of *v* is the distance to a farthest vertex from *v*. If $dist(u, v) = e(u)(v \neq u)$ we say that *v* is an eccentric vertex of *u*. We define $dist(u, v) = \infty$ whenever there is no path joining the vertices *u* and *v*. The *radius*, *rad(G)* and *diameter*, *diam(G)* are minimum and maximum eccentricities, respectively. A graph G is self-centered if $rad(G) = diam(G)$. An eccentric path of a vertex v is a geodesic from v to an eccentric vertex of v.

The distance degree sequence (dds) of a vertex v in a graph $G = (V, E)$ is a list of the number of vertices at distance $1, 2, \ldots, e(v)$ in that order, where $e(v)$ denotes the eccentricity of v in G. Thus, the sequence $(d_{i_0}, d_{i_1}, d_{i_2}, \ldots, d_{i_j}, \ldots)$ is the dds of the vertex v_i in G where, d_{i_j} denotes number of vertices at distance j from v_i. The concept of *distance degree regular (DDR)* graphs was introduced by G. S. Bloom et al. [17], as the graphs for which all vertices have the same dds. A graph is said to be an *unique eccentric node (u. e. n.)* graph if every vertex has a unique eccentric vertex. Moreover, if G is u. e. n. graph then, by Nandakumar and Parthasarathi [18] each vertex is eccentric.

Buckley [7] defined the *eccentric digraph* $ED(G)$ of a graph G as having the same vertex set as *G* and there is an arc from *u* to *v* if *v* is an eccentric vertex of *u*. In [7], Buckley has listed the eccentric digraphs of many classes of graphs including complete graphs, complete bipartite graphs, antipodal graphs and cycles and has given various interesting general structural properties of eccentric digraphs of graphs.

In [6], Akiyama et al. have defined *eccentric graph* of a graph *G*, denoted by G_e , has the same set of vertices as *G* with two vertices *u* and *v* being adjacent in G_e if and only if either *v* is an eccentric vertex of *u* in *G* or *u* is an eccentric vertex of *v* in *G*, that is $dist_G(u, v) = min\{e_G(u), e_G(v)\}$. Note that G_e is the underlying graph of $ED(G)$.

In [8], Boland and Miller introduced the concept of the *eccentric digraph* of a digraph. In [19], Gimbert et al. have proved that $G_e = ED(G)$ if and only if *G* is self-centered. In the same paper, the authors have characterized eccentric digraphs in terms of complement of the *reduction of G*, denoted by $\overline{G^-}$. Given a digraph *G* of order *p*, a reduction of *G*, denoted by G^-, is derived from *G* by removing all its arcs incident from vertices with out-degree $p - 1$. Note that $ED(G)$ is a subgraph of $\overline{G^-}$.

In [10], Gimbert et al. have studied on the behaviour of sequences of *iterated eccentric digraphs*. Given a positive integer $k \geq 2$, the k^{th} iterated eccentric digraph of G is written as $ED^k(G) = ED(ED^{k-1}(G))$, where $ED^0(G) = G$ and $ED^1(G) = ED(G)$. The iterated sequence of eccentric digraphs concerns with the smallest integer numbers $p > 0$ and $t \geq 0$ such that $ED^t(G) = ED^{p+t}(G)$. We call p the period of G and t the tail of G; these quantities are denoted $p(G)$ and $t(G)$ respectively. In [11], [8] Boland et al. have discussed many interesting results about eccentric digraphs. Also, they have listed open problems about these graphs.

2 Eccentric Digraph of Prism and Lexicographic Product

In this section we consider prism and lexicographic product of two graphs, which are defined as follows:

Definition 2.1. The prism of a graph G is defined as the cartesian product $G\square K_2$; that is, take two disjoint copies of G and add a matching joining the corresponding vertices in the two copies.

Remark 1. *If the distance between any two vertices in G is t, say $d(u, v) = t$, then in $G\square K_2$, $d(f(u), v) = t + 1$, where $f(u)$ is the mirror image of u.*

Remark 2. *For any graph G, degree of a vertex v is same in its eccentric digraph and eccentric digraph of $G\square K_2$.*

Remark 3. *If G is any (p, q) graph, then the eccentric digraph of $G\square K_2$ is $(2p, 2q)$ graph.*

Proposition 2.1. *For any graph G, if $ED(G)$ is disconnected then eccentric digraph of $G\square K_2$ is also disconnected.*

Proof. Let $ED(G)$ be disconnected having at least two components, say C_1 and C_2. In G no vertex of $V(C_1)$ has its eccentric vertex in $V(C_2)$ and vice-versa. In the prism $G\square K_2$, let G and $G^{'}$ be the two copies of G, having $V(C_1^{'}) \subset V(G^{'})$ and $V(C_2^{'}) \subset V(G^{'})$ as the mirror images of $V(C_1)$ and $V(C_2)$, respectively. In $G\square K_2$, using the distance in prism, no vertex of $V(C_1)$ has its eccentric vertex in $V(C_2^{'})$, and vice-versa, and no vertex of $V(C_2)$ has its eccentric vertex in $V(C_1^{'})$, and vice-versa. Hence $ED(G\square K_2)$ is disconnected. $\square$

Note: The converse of the above Proposition need not be always true. Consider the eccentric digraph of prism of K_2, which is a disconnected graph, but the eccentric digraph of K_2 is connected.

Theorem 2.1. *Eccentric digraph of a prism of an odd cycle C_p is isomorphic to an even cycle C_{2p}, that is, $ED(C_p\square K_2) \cong C_{2p}$.*

Proof. Let $C_p\square K_2$ be the prism of an odd cycle C_p and let $C_p^{'}$ and $C_p^{''}$ be two copies of C_p in the prism. Clearly $d(u', v') = l$ implies $d(u', v'') = l + 1$, where $u' \in C_p^{'}$ and $v'' \in C_p^{''}$ is an image of $v' \in C_p^{'}$ and $d(u'', v'') = l$ implies $d(u'', v') = l + 1$, where $u'' \in C_p^{''}$ and $v' \in C_p^{'}$ is an image of $v'' \in C_p^{''}$. Since C_p is an odd cycle, in G every vertex of $C_p^{'}$ has exactly two eccentric vertices in $C_p^{''}$ and vice versa. Hence $ED(C_p\square K_2)$ is a regular graph of regularity 2 on $2p$ vertices. If $ED(C_p\square K_2)$ is disconnected, then clearly C_p is disconnected, a contradiction. Therefore $ED(C_p\square K_2)$ is connected 2-regular graph. Hence eccentric digraph of a prism of an odd cycle C_p is isomorphic to an even cycle C_{2p}, given by $1, f(\frac{p+1}{2}), p, f(\frac{p+1}{2} - 1), p - 1, f(\frac{p+1}{2} - 2), p - 2, \ldots, f(\frac{p+1}{2} - \frac{p-1}{2}), p - \frac{p-1}{2}, f(p), p - \frac{p+1}{2}, f(p - 1), p - (\frac{p+1}{2} + 1), f(p - 2), p - (\frac{p+1}{2} + 2), \ldots, f(p - (\frac{p-3}{2})), 1$, where $f(1), f(2), \ldots, f(p)$ are the images of $1, 2, \ldots, p$, respectively in the prism. $\square$

Remark 4. *Prism of an u.e.n. DDR graph is of tail =1 and period = 2.*

As prism of an u.e.n. DDR graph is again u.e.n. DDR graph. Hence the result follows from the Proposition 3.3 in [12].
Note: In particular, prism of an even cycle C_p is of tail =1 and period = 2.

Lemma 2.1. *Prism of odd cycle C_p, $p \geq 3$ is of tail =2 and period = 2.*

Proof. Let C_p be an odd cycle, labeled as $1, 2, \ldots, p$ and G be the prism of C_p. Let $C_p^{'}$, labeled as $f(1), f(2), \ldots, f(p)$ be a copy of C_p in G. From Theorem 2.1, $ED(G)$ is an even cycle C_{2p} given by $1, f(\frac{p+1}{2}), p, f(\frac{p+1}{2}-1), p-1, f(\frac{p+1}{2}-2), p-2, \ldots, f(\frac{p+1}{2}-\frac{p-1}{2}), p-\frac{p-1}{2}, f(p), p-\frac{p+1}{2}, f(p-1), p-(\frac{p+1}{2}+1), f(p-2), p-(\frac{p+1}{2}+2), \ldots, f(p-(\frac{p-3}{2})), 1$, where $f(1), f(2), \ldots, f(p)$ are the images of $1, 2, \ldots, p$, respectively in the prism. Hence $ED(G)$ is u.e.n. graph. Clearly $ED^2(G)$ is the union of disjoint $K_2^{'}s$. $ED^3(G)$ is $2p-2$ regular u.e.n. graph and hence $ED^4(G)$ is again union of disjoint $K_2^{'}s$. Therefore $ED^2(G) = ED^4(G)$. Hence the proof. □

The lexicographic product is defined as follows:

Definition 2.2. Given graphs G and H, the lexicographic product $G[H]$ has vertex set $\{(g, h) : g \in V(G), h \in V(H)\}$ and two vertices (g, h), (g', h') are adjacent if and only if either $[g, g']$ is an edge of G or $g = g'$ and $[h, h']$ is an edge of H.

In [20], the distance between two vertices in the lexicographic product is given by,

$$d_{G[H]}((g,h),(g',h')) = \begin{cases} d_G(g,g'), & \text{if } g \neq g' \\ d_H(h,h'), & \text{if } g = g', \text{ and } d_G(g) = 0 \\ min\{d_H(h,h'), 2\}, & \text{if } g = g', \text{ and } d_G(g) \neq 0 \end{cases}$$

Theorem 2.2. *Eccentric digraph of prism of lexicographic product of an odd cycle C_p, $p \geq 7$ with a graph G, $C_p[G]$ is isomorphic to $C_{2p}[\overline{K}_m]$, where G is any graph of order m.*

Proof. Let C_p, $p \geq 7$ be an odd cycle and G be any graph of order m. Let $C_p[G]$ be lexicographic product of C_p with the graph G. We know from Theorem 2.1, eccentric digraph of prism of C_p is an even cycle C_{2p} given by $1, f(\frac{p+1}{2}), p, f(\frac{p+1}{2}-1), p-1, f(\frac{p+1}{2}-2), p-2, \ldots, f(\frac{p+1}{2}-\frac{p-1}{2}), p-\frac{p-1}{2}, f(p), p-\frac{p+1}{2}, f(p-1), p-(\frac{p+1}{2}+1), f(p-2), p-(\frac{p+1}{2}+2), \ldots, f(p-(\frac{p-3}{2})), 1$. Now consider the prism of $C_p[G]$ as shown in Figure 1. Using the distances defined in prism and lexicographic products and the above even cycle, we have the partition of vertex set of eccentric digraph of prism of $C_p[G]$, $G^1 \cup f(G^{\frac{p+1}{2}}) \cup G^p \cup f(G^{\frac{p+1}{2}-1}) \cup G^{p-1} \cup f(G^{\frac{p+1}{2}-2}) \cup G^{p-2} \ldots \cup f(G^{\frac{p+1}{2}-\frac{p-1}{2}}) \cup G^{p-\frac{p-1}{2}} \cup f(G^p) \cup G^{p-\frac{p+1}{2}} \cup f(G^{p-1}) G^{p-(\frac{p+1}{2}+1)} \cup f(G^{p-2}) \cup G^{p-(\frac{p+1}{2}+2)} \cup \ldots \cup f(G^{p-(\frac{p-3}{2})}) \cup G^1$, where each set contains independent vertices and the subgraph induced by any two consecutive sets is complete bipartite graphs and the subgraph induced by the nonconsecutive sets is totally disconnected. The eccentric digraph of prism of $C_p[G]$ is isomorphic to $C_{2p}[\overline{K}_m]$, as shown in Figure 2. □

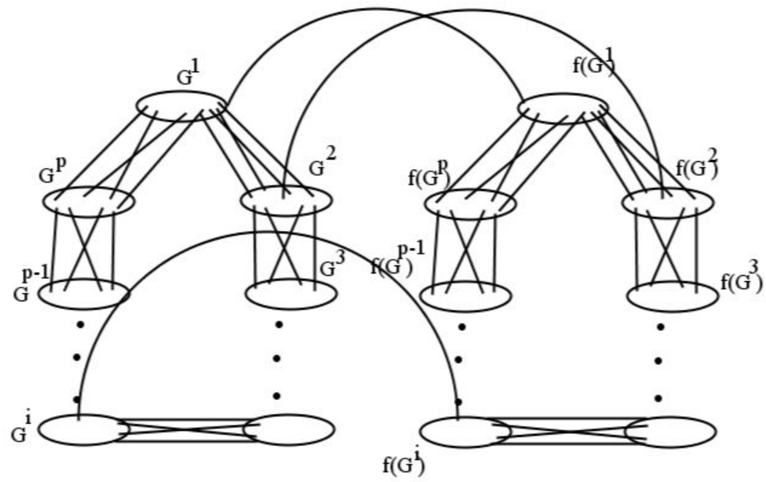

Figure 1: Prism of a $C_p[G]$

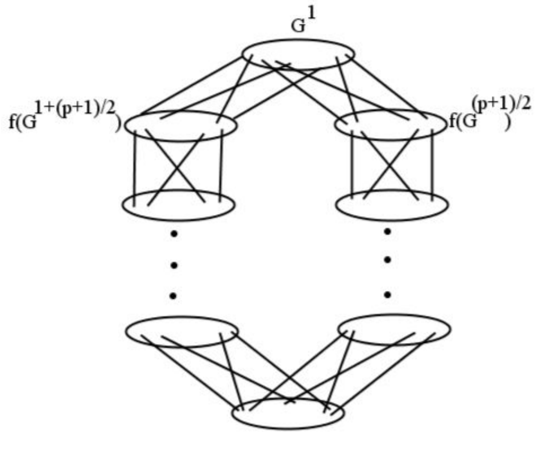

Figure 2: Eccentric digraph of prism of a lexicographic product

Corollary 2.1. *Prism of $C_p[G]$ is of period = 2 and tail = 3, where $p \geq 5$ is an odd integer.*

Proof. From Theorem 2.1, eccentric digraph of prism of $C_p[G]$ is isomorphic to $C_{2p}[\overline{K_m}]$. $ED^2(C_p[G]\square K_2)$ is the union of p complete bipartite graphs $K_{m,m} \cup K_{m,m} \cup \ldots \cup K_{m,m}$. $ED^3(C_p[G]\square K_2)$ contains the union of p sets $S_1 \cup S_2 \cup \ldots \cup S_p$ each of order $2m$, such that each set contains independent vertices and every vertex in each S_i is adjacent to all vertices in $\cup_{j \neq i}^{p} S_j$. $ED^4(C_p[G]\square K_2)$ is the union of complete graphs $K_{2m} \cup K_{2m} \cup \ldots \cup K_{2m}$. $ED^5(C_p[G]\square K_2)$ contains the union of p sets $T_1 \cup T_2 \cup \ldots \cup T_p$ each of order $2m$, such that each set contains independent vertices and every vertex in each T_i is adjacent to all vertices in $\cup_{j \neq i}^{p} T_j$. Hence $ED^3(C_p[G]\square K_2) \cong ED^5(C_p[G]\square K_2)$. Hence the proof. $\square$

Theorem 2.3. *Eccentric digraph of $C_{2k}[G]\Box K_2$ is isomorphic to disjoint union of $2k$ number of complete bipartite graphs $K_{n,n}$, where n is the order of G.*

Proof. Let $C_{2k}[G]$ be a lexicographic product of an even cycle C_{2k} with a graph G as shown in Figure 3.

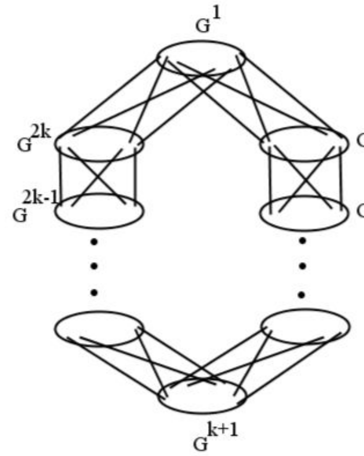

Figure 3: Lexicographic product of an even cycle with G

Since C_{2k} is an even cycle and from the distance in lexicographic product, for each G^i there exists exactly one G^j such that each vertex in G^i is eccentric to every vertex in G^j. Hence for each G^i there exists G^{k+i} and vice versa.
The structure of the prism of $C_{2k}[G]$ is as shown in Figure 4.

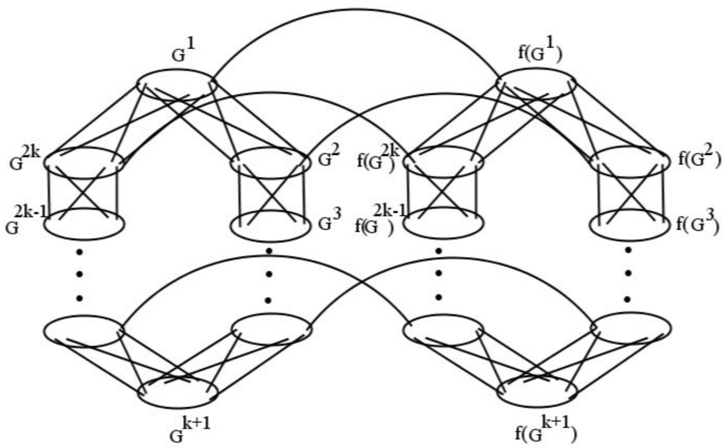

Figure 4: Prism of lexicographic product of an even cycle with G

In Figure 4, each $f(G^i)$ is the mirror image of G^i and each vertex in G^i is adjacent to its image in $f(G^i)$. Hence by Remark 1 each vertex in G^i is eccentric to every vertex of $f(G^{k+i})$ and vice-versa. Since the prism of $C_{2k}[G]$ contains $2k$ number of $G^{i'}s$ and $2k$ number of $f(G^i)'s$, the eccentric digraph of prism of $C_{2k}[G]$ is the union of $2k$ complete bipartite graphs. Hence the proof. □

Corollary 2.2. *Eccentric digraph of prism of $C_k[\overline{K}_m]$ is isomorphic to $C_{2k}[\overline{K}_m]$.*

Corollary 2.3. *Prism of $C_{2k}[G]$ is of period = 2 and tail = 2, where $k \geq 4$ is an even integer.*

Proof. From Theorem 2.3, eccentric digraph of $C_{2k}[G]\Box K_2$ is the disjoint union of $2k$ number of complete bipartite graphs $K_{n,n}$, where n is the order of G. $ED^2(C_{2k}[G]\Box K_2)$ contains the union of $2k$ sets $S_1 \cup S_2 \cup \ldots \cup S_{2k}$ each of order $2n$, such that each set contains independent vertices and every vertex in each S_i is adjacent to all vertices in $\cup_{j\neq i}^{2k} S_j$. $ED^3(C_{2k}[G]\Box K_2)$ is the union of complete graphs $K_{2n} \cup K_{2n} \cup \ldots \cup K_{2n}$. $ED^4(C_{2k}[G]\Box K_2)$ contains the union of $2k$ sets $T_1 \cup T_2 \cup \ldots \cup T_{2k}$ each of order $2n$, such that each set contains independent vertices and every vertex in each T_i is adjacent to all vertices in $\cup_{j\neq i}^{2k} T_j$. Hence $ED^2(C_{2k}[G]\Box K_2) \cong ED^4(C_{2k}[G]\Box K_2)$. Hence the proof. □

3 Eccentric Digraph of Cartesian Product of Two Graphs

Here we consider the cartesian product of graphs and results on eccentric digraphs of cartesian products of graphs.

Definition 3.1. The cartesian product of two graphs G and H, denoted by $G\Box H$, is a graph with vertex set $V(G\Box H) = V(G) \times V(H)$, that is, the set $\{(g,h)/g \in G, h \in H\}$.

The edge set of $G\Box H$ consists of all pairs $[(g_1,h_1),(g_2,h_2)]$ of vertices with $[g_1,g_2] \in E(G)$ and $h_1 = h_2$, or $g_1 = g_2$ and $[h_1,h_2] \in E(H)$.

In the cartesian product of any two graphs, the distance between any two vertices (u_1,v_1) and (u_2,v_2) is given by $d_{G_1\Box G_2}((u_1,v_1),(u_2,v_2)) = d_{G_1}(u_1,u_2) + d_{G_2}(v_1,v_2)$ as in [21].

Remark 5. *Degree of (u,v) in $ED(G_1\Box G_2)$ is the product of degree of u in $ED(G_1)$ and degree of v in $ED(G_2)$. Since, the number of eccentric vertices of (u,v) in $G_1\Box G_2$ is the product of the number of eccentric vertices of u in G_1 and the number of eccentric vertices of v in G_2.*

Proposition 3.1. *For any two self centered graphs G_1 and G_2, $ED(G_1\Box G_2)$ is regular if and only if both $ED(G_1)$ and $ED(G_2)$ are regular.*

Proof. Let $ED(G_1\Box G_2)$ be regular. Suppose, on the contrary, if one of $ED(G_1)$ or $ED(G_2)$ is not regular, then three cases arise.
Case(i): $ED(G_1)$ is not regular and $ED(G_2)$ is regular.
Case(ii): $ED(G_1)$ is regular and $ED(G_2)$ is not regular.
Case(iii): Both $ED(G_1)$ and $ED(G_2)$ are not regular.
Case(i): Let $ED(G_1)$ be not regular and $ED(G_2)$ be regular. There exist at least two vertices u and v in G_1 having k_1 and $k_2(\neq k_1)$ number of eccentric vertices, respectively and since $ED(G_2)$ is regular, the number of eccentric vertices of every vertex remains the same, say k. Using distance in cartesian product, the number of eccentric vertices of (u,w) and (v,w) is kk_1 and kk_2 respectively. Hence $ED(G_1\Box G_2)$ is not regular, a contradiction proves the result.
Case(ii): Proof follows on the similar lines as in Case(i).
Case(iii): Since no graph is perfect, in $ED(G_1)$, there exist at least two vertices, say u and v having the same degree, i.e., the number of eccentric vertices from both u and v in G_1 is the same and let x and y be any two vertices in $ED(G_2)$ having different degrees, i.e., the number of eccentric vertices from both x and y in G_2 is different. Let A_u and A_v be the set of eccentric vertices of u and v respectively in G_1 and A_x and A_y be the set of eccentric vertices of x and y respectively in G_2.

Using distance in cartesian product, $A_u \times A_x$ is the set of vertices eccentric to (u, x) in $G_1 \Box G_2$ and $A_v \times A_y$ is the set of vertices eccentric to (v, y) in $G_1 \Box G_2$. Since the cardinalities of A_u and A_v are the same and the cardinalities of A_x and A_y are different, the cardinalities of $A_u \times A_x$ and $A_v \times A_y$ are different. Hence $deg_{ED(G_1 \Box G_2)}(u, x) \neq deg_{ED(G_1 \Box G_2)}(v, y)$. Hence $ED(G_1 \Box G_2)$ is not regular.
Now suppose, both $ED(G_1)$ and $ED(G_2)$ are regular with regularities k_1 and k_2, i.e., the number of eccentric vertices of each vertex in G_1 and G_2 is k_1 and k_2 respectively. Hence, the number of eccentric vertices of each vertex in $G_1 \Box G_2$ is $k_1 k_2$, making $ED(G_1 \Box G_2)$ regular with regularity $k_1 k_2$. □

Corollary 3.1. *If there exist at least two vertices u, v in G_1 such that the number of eccentric vertices of u and v are not same and the number of eccentric vertices of every vertex in G_2 is k then $ED(G_1 \Box G_2)$ is irregular.*

Proposition 3.2. *If G_1 is any connected non self centered graph and G_2 is self centered u.e.n. graph then eccentric digraph of $G_1 \Box G_2$ is the disjoint union of bipartite digraphs.*

Proof. Let G_1 be any connected non self centered graph with the vertex set $V(G_1) = \{x_1, x_2, \ldots, x_m\}$ and G_2 be self centered u.e.n. graph with the vertex set $V(G_2) = \{y_1, y_2, \ldots, y_n\}$. Let $G_1 \Box G_2$ be the cartesian product of G_1 and G_2. In $G_1 \Box G_2$, (x_i, y_j) is the eccentric vertex of (x_r, y_t), if x_i is the eccentric vertex of x_r and y_j is the eccentric vertex of y_t. Since G_2 is u.e.n. self centered graph, whenever y_j is the eccentric vertex of y_t, each vertex of $S_t = V(G_1) \times y_t = \{(x_i, y_t)/x_i \in V(G_1)\}$ has the eccentric vertex in $S_j = V(G_1) \times y_j = \{(x_i, y_j)/x_i \in V(G_1)\}$ only. There exist $\frac{|V(G_2)|}{2}$ such pairs of sets. Hence, the $ED(G_1 \Box G_2)$ is disconnected having $\frac{|V(G_2)|}{2}$ components and each component is bipartite graph, since each set contains no two vertices such that one is eccentric to other. Also, since G_1 is non self centered graph, there exists at least one pair of vertices (x_f, y_k) and (x_d, y_l) such that (x_f, y_k) is eccentric to (x_d, y_l) but (x_d, y_l) is not eccentric to (x_f, y_k). Hence $ED(G_1 \Box G_2)$ is a digraph. Hence the proof. □

Remark 6. *In Proposition 3.2, each component of $ED(G_1 \Box G_2)$ is isomorphic to eccentric digraph of prism of G_1.*

Remark 7. *If G_1 is a self centered graph and G_2 is self centered u.e.n graph then eccentric digraph of $G_1 \Box G_2$ is disjoint union of bipartite graphs. In particular, if G_1 is an odd cycle C_n and G_2 is an even cycle C_m, $n > m \geq 4$, then $ED(C_m \Box C_n)$ is the disjoint union of $\frac{m}{2}$ number of even cycles each of length $2n$.*

Remark 8. *If G_1 and G_2 are self centered u.e.n. graphs then eccentric digraph of $G_1 \Box G_2$ is disjoint union of $K_2^{'}s$.*

4 Eccentric Digraphs of Normal Product of Two Graphs

Here we consider the normal product of graphs and results on eccentric digraphs of normal products of graphs.

Definition 4.1. The normal product of two graphs G and H, denoted $G \oplus H$, is a graph with vertex set $V(G \oplus H) = V(G) \times V(H)$, that is, the set $\{(g, h)/g \in G, h \in H\}$, and an edge $[(g_1, h_1), (g_2, h_2)]$ exists whenever any of the following conditions holds:
(i) $[g_1, g_2] \in E(G)$ and $h_1 = h_2$,
(ii) $g_1 = g_2$ and $[h_1, h_2] \in E(H)$,
(iii) $[g_1, g_2] \in E(G)$ and $[h_1, h_2] \in E(H)$.

Stevanovic' [3] has considered the distance between any pair of vertices in normal product. Given two vertices (u_i, v_j) and (u_k, v_m), the distance between these two vertices in the normal product is given by:

$$d_{G_1 \oplus G_2}((u_i, v_j), (u_k, v_m)) = max\{d_{G_k}(u_i, v_k), d_{G_i}(u_j, v_m)\}$$

Theorem 4.1. *If G_1 and G_2 are any two graphs of order p_1 and p_2, respectively, then the number of eccentric vertices of (u, v) in the normal product $G_1 \oplus G_2$ is given by*

$$k = \begin{cases} p_1 \cdot k_2, & \text{if } e(u) < e(v) \\ p_2 \cdot k_1, & \text{if } e(u) > e(v) \\ p_1 \cdot k_2 + p_2 \cdot k_1 - k_1 \cdot k_2, & \text{if } e(u) = e(v) \end{cases}$$

where k is the number of eccentric vertices of (u, v) in $G_1 \oplus G_2$, k_1 and k_2 are the number of eccentric vertices of $u \in G_1$ and $v \in G_2$, respectively.

Proof. Let G_1 and G_2 be any two graphs of orders p_1 and p_2, respectively. Let (u, v) be any vertex in $G_1 \oplus G_2$, where $u \in G_1$ and $v \in G_2$. Let $u \in G_1$ has k_1 eccentric vertices and $v \in G_2$ has k_2 eccentric vertices. To obtain the number of eccentric vertices of (u, v) in $G_1 \oplus G_2$, three cases arise: Case(i): $e(u) < e(v)$, Case(ii): $e(u) > e(v)$ and Case(iii): $e(u) = e(v)$.
Case(i): Let $e(u) < e(v)$. Let $Y \subseteq V(G_2)$ be the set of vertices eccentric to v. From the definition of distance in normal product [3], $V(G_1) \times Y = \{(x, y)/x \in V(G_1), y \in Y\}$ is the set of vertices eccentric to (u, v) in $G_1 \oplus G_2$, hence the number of eccentric vertices of (u, v) in $G_1 \oplus G_2$ is $p_1 \cdot k_2$.
Case(ii): Let $e(u) > e(v)$. Let $X \subseteq V(G_1)$ be the set of vertices eccentric to u. Hence, $X \times V(G_2) = \{(x, y)/x \in X, y \in V(G_2)\}$ is the set of vertices eccentric to (u, v) in $G_1 \oplus G_2$. So, the number of eccentric vertices of (u, v) in $G_1 \oplus G_2$ is $p_2 \cdot k_1$.
Case(iii): Let $e(u) = e(v)$. Let $X \subseteq V(G_1)$ be the set of vertices eccentric to u and $Y \subseteq V(G_2)$ be the set of vertices eccentric to v. Clearly $\{X \times V(G_2) \cup V(G_1) \times Y\} \setminus \{X \times Y\}$ is the set of vertices eccentric to (u, v) in $G_1 \oplus G_2$. Hence, the number of eccentric vertices of (u, v) in $G_1 \oplus G_2$ is $p_1 \cdot k_2 + p_2 \cdot k_1 - k_1 \cdot k_2$. □

Theorem 4.2. *Let G_1 be a self centered graph of order p_1, such that $ED(G_1)$ is regular with regularity k and G_2 be any graph of order p_2 such that $diam(G_1) \geq diam(G_2)$, then $ED(G_1 \oplus G_2)$ is regular with regularity $k \cdot p_2$.*

Proof. Let G_1 be a self centered graph such that $ED(G_1)$ is regular with regularity k, hence in G_1, the number of eccentric vertices of every vertex remains same. Let (u, v) be any vertex in $G_1 \oplus G_2$ and S_1 be a set of eccentric vertices of u in G_1. In [3], the distance in normal product is defined as $d[(u_i, v_j), (u_m, v_n)] = max\{d[u_i, u_m], d[v_j, v_n]\}$, hence $S_1 \times V(G_2)$ is the set of eccentric vertices of (u, v) in $G_1 \oplus G_2$, since $diam(G_1) \geq diam(G_2)$ and G_1 is a self centered graph. Hence $deg(u, v)$ in $ED(G_1 \oplus G_2)$ is $|S_1| \cdot |V(G_2)| = k \cdot |V(G_2)|$. Since, the vertex (u, v) is arbitrarily chosen, the graph $ED(G_1 \oplus G_2)$ is regular with regularity $k \cdot p_2$. □

Theorem 4.3. *Eccentric digraph of normal product of an even cycle C_n and any cycle C_m, where $m < n$ is the disjoint union of $\frac{n}{2}$ complete bipartite graphs $K_{m,m}$.*

Proof. Let $C_n : u_1, u_2, \ldots, u_n$ be an even cycle and $C_m : v_1, v_2, \ldots, v_m$ be any cycle, where $m < n$, hence $diam(C_m) < diam(C_n)$. Using distance in normal product [3], for every pair of vertices u_i, u_t eccentric to each other in C_n, there exist two sets of vertices $S_i = \{(u_i,\ v_j), 1 \leq j \leq m\}$ and $S_t = \{(u_t,\ v_j), 1 \leq j \leq m\}$ in $C_m \oplus C_n$ such that each vertex of S_1 is eccentric to every vertex of S_2 and vice - versa, as shown in Figure 5. Hence, the subgraph of $ED(C_m \oplus C_n)$ induced by these two sets forms a complete bipartite graph $K_{m,m}$. Since C_n is u.e.n graph, it has exactly $\frac{n}{2}$ such pairs. Hence $ED(C_m \oplus C_n)$ has $\frac{n}{2}$ disjoint complete bipartite graphs $K_{m,m}$. Hence the proof. □

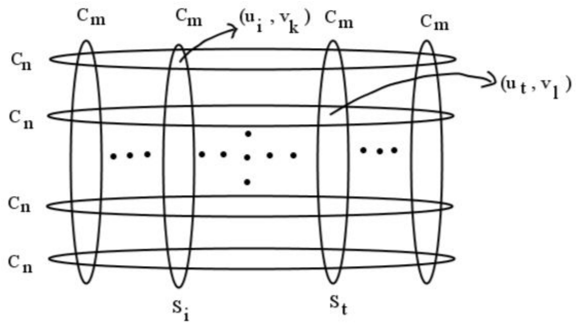

Figure 5: Normal product of even cycle and any cycle

Theorem 4.4. *Let G_1 and G_2 be any two connected graphs such that $diam(G_1) < rad(G_2)$ then the eccentric digraph of normal product of G_1 and G_2 is isomorphic to $G[\overline{K}_m]$, where $G \cong ED(G_2)$ and m is the order of G_1.*

Proof. Let G_1 and G_2 be any two connected graphs having orders m and n, respectively. Let $V(G_1) = \{x_1, x_2, x_3, \ldots, x_m\}$, $V(G_2) = \{y_1, y_2, y_3, \ldots, y_n\}$ and $S_j = \{(x_i, y_j)/x_i \in G_1\}$, $1 \leq j \leq n$ be sets of vertices in the normal product $G_1 \oplus G_2$. Using the distance in normal product and $diam(G_1) < rad(G_2)$, it is clear that the vertex (x, y) is eccentric to $(x^{'}, y^{'})$ if and only if y is eccentric to $y^{'}$. Hence, each vertex of S_{k_1} is eccentric to every vertex of S_{k_2}, whenever y_{k_1} is eccentric to y_{k_2} and each S_j is an independent set. Hence $ED(G_1 \oplus G_2)$ is isomorphic to the lexicographic product $G[\overline{K}_m]$, where $G \cong ED(G_2)$. □

Remark 9. *Eccentric digraph of normal product of an odd cycle C_n and a cycle C_m is isomorphic to $C_n[\overline{K}_m]$, $n > m \geq 3$, if m is odd and $n > m + 1 \geq 5$, if m is even.*

Remark 10. *Let G be a connected graph and C_n be an odd cycle such that $diam(G) < diam(C_n)$ then the eccentric digraph of normal product of G and C_n is $C_n[\overline{K}_m]$.*

Remark 11. *Eccentric digraph of normal product of K_m and C_n is isomorphic to $C_n[\overline{K}_m]$, where $n \geq 5$ is odd and $m \geq 1$ is any integer.*

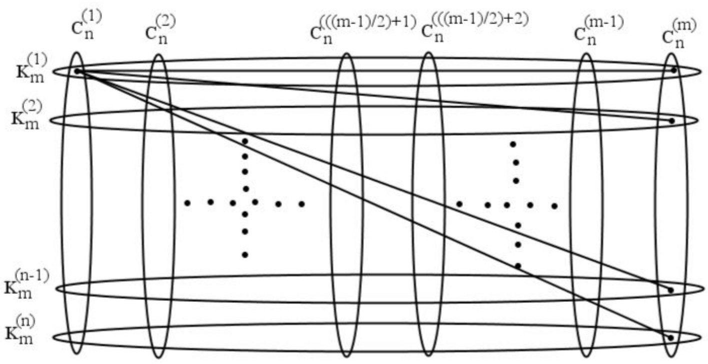

Figure 6: Normal product of cycle and complete graph

Proposition 4.1. *Let G_1 and G_2 be any two connected graphs such that $diam(G_1) < rad(G_2)$ then the period and tail of $G_1 \oplus G_2$ are same as that of G_2.*

Proof. Let G_1 and G_2 be any two connected graphs such that $diam(G_1) < rad(G_2)$. Let $G_1 \oplus G_2$ be the normal product of G_1 with G_2. In $G_1 \oplus G_2$, the vertex (x, y) is eccentric to (x', y') if and only if x is eccentric to x', since $e(u) < e(v)$ for all $u \in V(G_1)$ and $v \in V(G_2)$. Hence $ED(G_1 \oplus G_2) \cong ED(G_2[\overline{G_1}]) \cong ED(G_2)[\overline{G_1}]$. $ED^n(G_1 \oplus G_2) \cong ED^n(G_2[\overline{G_1}]) \cong ED^n(G_2)[\overline{G_1}]$, where n is any positive integer. Hence the period and tail of $G_1 \oplus G_2$ is same as that of G_2. ☐

Proposition 4.2. *If G_1 is a disconnected graph having k components and G_2 is a connected graph then the eccentric digraph of normal product of G_1 and G_2 is a complete k-partite graph $K_{n_1 m, n_2 m, \ldots n_k m}$, where m is the order of G_2 and n_i, $1 \leq i \leq k$ is the order of each component of G_1.*

Proof. Let G_1 be a disconnected graph having k components and G_2 be a connected graph. Clearly, the normal product $G_1 \oplus G_2$ is a disconnected graph having k components. Hence the eccentric digraph of $G_1 \oplus G_2$ is a complete k-partite graph. ☐

Corollary 4.1. *If G_1 and G_2 are two disconnected graphs having k_1 and k_2 components respectively, then the eccentric digraph of normal product of G_1 and G_2 is complete $k_1 \cdot k_2$-partite graph.*

Remark 12. *There exist no two graphs G_1 and G_2 such that $ED(G_1 \Box G_2) = ED(G_1 \oplus G_2)$ because the number of vertices eccentric to any vertex (u, v) in $G_1 \Box G_2$ is always greater than the number of vertices eccentric to (u, v) in $G_1 \oplus G_2$.*

Remark 13. *There exist no two graphs G_1 and G_2 such that*
$ED(G_1 \oplus G_2) = G_1 \oplus G_2$.

5 Conclusions

Gimbert et al. [9] had posed a conjecture in the year 2005 on the period of cartesian product of two odd cycles as follows:
Conjecture: $p(C_{2m+1} \times C_{2m+1}) = p(C_{2m+1}) + p(C_{2m+1})$, where $\times$ denotes the usual Cartesian product of graphs.
Settling this Conjecture seems difficult at this point of time, but the results discussed in this paper serve as stepping stones for further research in this direction.

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Competing Interests

The authors declare that no competing interests exist.

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