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Research Article

Phase Structure and Quasinormal Modes of AdS Black Holes in Rastall Theory

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We discuss the $P-V$ criticality and phase transition in the extended phase space of anti-de Sitter(AdS) black holes in four-dimensional Rastall theory and recover the Van der Waals (VdW) analogy of small/large black hole (SBH/LBH) phase transition when the parameters ω_s and ψ satisfy some certain conditions. Later, we further explore the quasinormal modes (QNMs) of massless scalar perturbations to probe the SBH/LBH phase transition. It is found that it can be detected near the critical point, where the slopes of the QNM frequencies change drastically in small and large black holes.

1. Introduction

As the most beautiful and simplest theory of gravity, Einstein's general relativity (GR) admits the covariant conservation of matter energy-momentum tensor. It is worthy to point out that the idea of the covariant conservation for spacetime symmetries has been implemented only in the Minkowski flat or weak field regime of gravity. Nevertheless, the actual nature of the spacetime geometry and the covariant conservation relation is still debated in the strong domain of gravity. In 1972, Rastall [1] demonstrated an adjustment to Einstein's equation, which results in a violation of the usual conservation law, and the energy-momentum tensor satisfies

$$T^{\mu\nu}{}_{;\mu} = \lambda R^{\nu}, \tag{1}$$

where R is the Ricci scalar and λ is the Rastall coupling parameter, which measures the potential deviation of Rastall theory from GR. This theory provides an explanation of the inflation problem, as the simplest modified gravity scenario to realize the late-time acceleration and other cosmological problems [2–5]. It is an interesting result that all electrovacuum solutions of GR automatically meet the equation of motion in the Rastall gravity. However, it failed if one introduces any nonvanishing trace matter field. Until now, many works on the various black hole solutions have been investigated in Rastall theory. The spherically symmetric black hole solutions were constructed in Refs. [6–10], the rotating black holes were in Refs. [11, 12], the thermodynamics of black holes was in Refs. [13–17], and also instability of black holes was in Refs. [18, 19].

Recently, the study of thermodynamics of AdS black holes has been generalized to the extended phase space, where the cosmological constant is related to the thermodynamic pressure [20, 21].

$$P = \frac{\Lambda}{8\pi}. \tag{2}$$

In fact, the variation of the cosmological constant is beneficial to the consistency between the first law of black hole thermodynamics and the Smarr formula. In the extended phase space, the charged AdS black hole admits a more direct and precise coincidence between the first-order small/large black hole (SBH/LBH) phase transition and Van der Waals (VdW) liquid-gas phase transition, and both systems share the same critical exponents near the critical point [22]. More discussions in this direction can be found as well in including reentrant phase transitions and some other phase transitions [23–51].

On the other hand, in the context of the AdS/CFT correspondence, the QNMs of a $(D+1)$-dimensional asymptotically AdS black hole or brane are poles of the retarded Green's function in the D-dimensional dual conformal field theory at strong coupling [52–54]. Then, one can describe various properties of strongly coupled quark-gluon plasmas which cannot be studied by traditional perturbative methods of quantum field theory [55, 56], such as the universal value $1/4\pi$ for the ratio of viscosity to the entropy density in quark-gluon plasma via various gravitational backgrounds [57]. In the dual-field theory, thermodynamic phase transition of black holes corresponds to the onset of instability of a black hole. It is naturally considered that QNMs of black holes are connected with thermodynamic phase transitions of strongly coupled field theories [58]. A lot of discussions have been focused on this topic, and more and more evidence has been found between thermodynamical phase transitions and QNMs [50, 59–70]. Recently, the extended phase space thermodynamics for $P-V$ criticality and phase transition of d-dimensional AdS black holes in perfect fluid background have been investigated in Ref. [17], which shows the existence of Van der Waals analogy of SBH/LBH phase transition. Motivated by the result, in this paper, we use the QNM frequencies of a massless scalar perturbation to probe the Van der Waals-like SBH/LBH phase transition of four-dimensional AdS black holes surrounded by perfect fluid in the Rastall theory.

This paper is organized as follows. In Section 2, we review the thermodynamics of four-dimensional AdS black holes in the extended phase space and will show the analogy of the SBH/LBH phase transition with the VdW liquid-gas system. In Section 3, we will disclose that the phase transition can be reflected by the QNM frequencies of dynamical perturbations. We end the paper with conclusions and discussions in Section 4.

2. Thermodynamics and Phase Transition of AdS Black Holes

Considering (1), the field equation including the negative cosmological constant Λ reads as [17]

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = k\left(T_{\mu\nu} - \lambda g_{\mu\nu} R\right), \tag{3}$$

where these field equations reduce to GR field equations in the limit of $\lambda \to 0$, κ equals to $8\pi G_N$, and G_N is the Newton gravitational coupling constant.

In four-dimensional spacetime, the energy-momentum tensor $T_{\mu\nu}$ of perfect fluid reads as [8, 9]

$$\begin{aligned} \mathcal{T}^{u}{}_{u} &= \mathcal{T}^{r}{}_{r} = -\rho_s(r), \\ \mathcal{T}^{\theta}{}_{\theta} &= \mathcal{T}^{\varphi}{}_{\theta_{d-2}} = \frac{1}{2}(1+3\omega_s)\rho_s(r). \end{aligned} \tag{4}$$

Then, the AdS black hole solution in four-dimensional Rastall theory is [17]

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\vartheta^2\right), \tag{5}$$

$$f(r) = 1 - \frac{2M}{r} - \frac{N_s}{r\xi} + \frac{r^2}{R_\Lambda^2}, \tag{6}$$

with

$$\begin{aligned} \xi &= \frac{1+3\omega_s - 6\psi(1+\omega_s)}{1-3\psi(1+\omega_s)}, \quad \psi = k\lambda, \\ \frac{1}{R_\Lambda^2} &= -\frac{\Lambda}{3(1-4\psi)}, \end{aligned} \tag{7}$$

where R_Λ is the curvature radius in the Rastall gravity and ω_s is the state parameter of fluid. M and N_s are two integration constants representing the black hole mass and surrounding field structure parameter, respectively. The subscript "s" denotes the surrounding field, like the dust, radiation, quintessence, cosmological constant, or phantom field.

Moreover, the integration constant N_s is related to the energy density ρ_s [9, 17]:

$$\rho_s = -\frac{3W_s N_s}{\kappa r^{(3(1+\omega_s)-12\psi(1+\omega_s))/(1-3\psi(1+\omega_s))}}, \tag{8}$$

with

$$W_s = -\frac{(1-4\psi)(\psi(1+\omega_s)-\omega_s)}{(1-3\psi(1+\omega_s))^2}. \tag{9}$$

Regarding the weak energy condition representing the positivity of any kind of energy density of the surrounding field, i.e., $\rho_s \geq 0$, the following condition was imposed:

$$W_s N_s \leq 0, \tag{10}$$

which implies that for the surrounding field with geometric parameter $W_s > 0$, we need $N_s < 0$ and conversely for $W_s < 0$, we need $N_s > 0$ [9]. When N_s vanishes, (6) reduces to vacuum AdS black hole solution in the Rastall gravity:

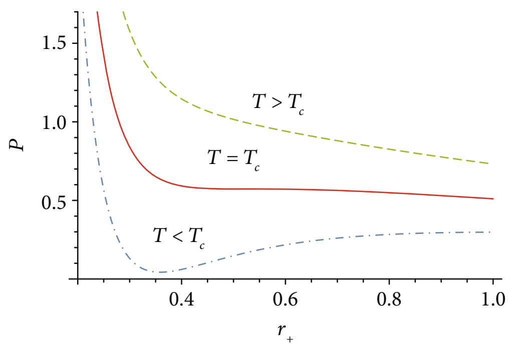
$$f(r) = 1 - \frac{2M}{r} + \frac{r^2}{R_\Lambda^2}. \tag{11}$$

In the limit of $\psi \to 0$, namely, $\lambda \to 0$, the covariant derivative of energy-momentum tensor vanishes and the Rastall gravity becomes the Einstein gravity. We can recover the Schwarzschild-AdS black hole from (3):

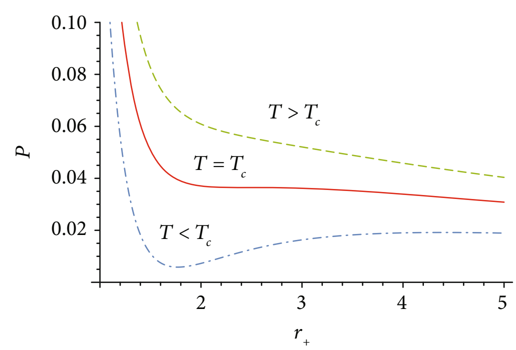
$$\begin{aligned} f(r) &= 1 - \frac{2M}{r} + \frac{r^2}{R^2}, \\ \frac{1}{R^2} &= -\frac{\Lambda}{3}. \end{aligned} \tag{12}$$

Table 1: The positive values of critical points in the case of $\rho_s \geq 0$.

$N_s > 0 \cup W_s < 0$	$\omega_s < -1 \cup \frac{1}{3+3\omega_s} < \psi < \frac{1}{4}$	$-1 < \omega_s < \frac{1}{3} \cup \frac{\omega_s}{1+\omega_s} < \psi < \frac{1+3\omega_s}{6+6\omega_s}$	None
$N_s > 0 \cup W_s < 0$	$\omega_s < -1 \cup \frac{1}{4} < \psi < \frac{2+3\omega_s}{9+9\omega_s}$	$-1 < \omega_s \leq \frac{1}{3} \cup \psi < \frac{\omega_s}{1+\omega_s}$	$\omega_s > \frac{1}{3} \cup \psi < \frac{1}{3+3\omega_s}$



(a) $\omega_s = -2/3$



(b) $\omega_s = 1/3$

Figure 1: The $P - r_+$ diagrams of four-dimensional AdS black holes with $\psi = -5/2$ and $N_s = -1$. The upper dashed line corresponds to the idea gas phase behavior for $T > T_c$. The critical temperature case $T = T_c$ is denoted by the solid line. The line below is with temperatures smaller than the critical temperature. We have $T_c = 0.1610$ in (a) and $T_c = 0.0433$ in (b).

In terms of horizon radius r_+, mass M, Hawking temperature T, and entropy S of the AdS black holes can be written, respectively, as

$$\begin{aligned} M &= \frac{r_+}{2}\left(1 - \frac{N_s}{r_+^{\xi}} + \frac{r_+^2}{R_\Lambda^2}\right), \\ T &= \frac{1}{4\pi}\left(\frac{1}{r_+} + \frac{3r}{R_\Lambda^2} + \frac{(\xi-1)N_s}{r_+^{1+\xi}}\right), \\ S &= \pi r_+^2. \end{aligned} \tag{13}$$

In the extended phase space, the cosmological constant is related to the thermodynamic pressure with

$$P = -\frac{\Lambda}{8\pi} = \frac{3(1-4\psi)}{8\pi R_\Lambda^2}, \tag{14}$$

we can obtain the equation of state:

$$P = \frac{(1-4\psi)}{2r_+}\left[T - \frac{1}{4\pi r_+} - \frac{(\xi-1)N_s}{4\pi r_+^{1+\xi}}\right], \tag{15}$$

from the Hawking temperature (13).

As usual, a critical point occurs when P has an inflection point:

$$\left.\frac{\partial P}{\partial r_+}\right|_{T=T_c, r_+=r_c} = \left.\frac{\partial^2 P}{\partial r_+^2}\right|_{T=T_c, r_+=r_c} = 0. \tag{16}$$

The corresponding critical values are obtained as

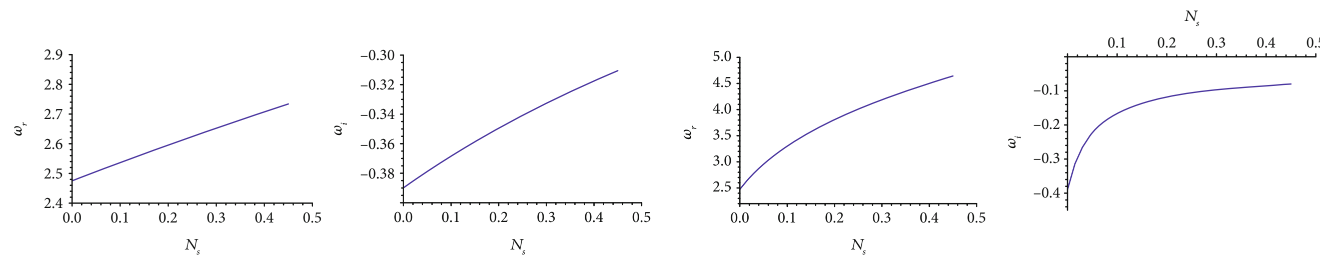
$$\begin{aligned} r_c &= \left[\frac{(1-\xi)(1+\xi)(2+\xi)N_s}{2}\right]^{1/\xi}, \\ T_c &= \frac{\xi}{2\pi(1+\xi)r_c}, \\ P_c &= \frac{(1-4\psi)\xi}{8\pi r_c^2(2+\xi)}. \end{aligned} \tag{17}$$

The subscript "c" denotes the values of the physical quantities at the critical points. Evidently, the critical parameters r_c, T_c, and P_c depend on the values of ω_s (in [7]), N_s, and ψ. Regarding the weak energy condition $\rho_s > 0$, we summarize the corresponding constraint conditions of the positive critical values r_c, T_c, and P_c in Table 1.

For instance, we plot the $P - r$ isotherm diagram for the quintessence surrounding field ($\omega_s = -2/3$) [71] and radiation surrounding field ($\omega_s = 1/3$) [72] in the region of $-1 < \omega_s \leq 1/3 \cup \psi < \omega_s/(1+\omega_s)$ in Figure 1. The dotted line corresponds to the "idea gas" phase behavior when $P > P_c$, and the VdW-like SBH/LBH phase transition appears in the system when $P < P_c$.

The behavior of the Gibbs free energy G is important to determine the thermodynamic phase transition. The free energy G obeys the following thermodynamic relation $G = M - TS$ with

$$G = \frac{r_+}{4} - \frac{2P\pi r_+^3}{3(1-4\psi)} - \frac{(1+\xi)N_s}{4r_+^{\xi-1}}. \tag{18}$$



(a) $\omega_s = -2/3$ (b) $\omega_s = 1/3$

FIGURE 2: For $n = 0$, (a) real quasinormal frequency and (b) negative imaginary quasinormal frequency as functions of N_s around the AdS black hole with $r_+ = 0.2$, $R_\Lambda = 1$, and $\psi = -5/2$ in the Rastall gravity.

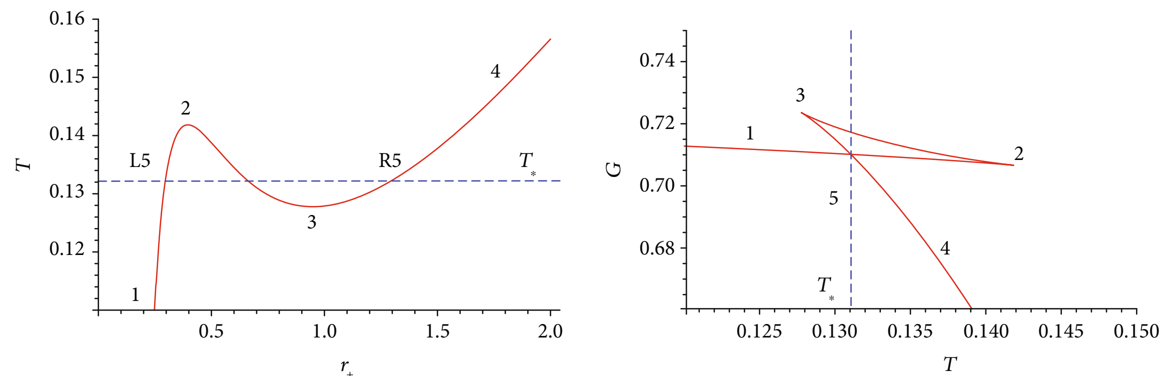


FIGURE 3: The $T - r_+$ and $G - T$ diagrams of four-dimensional AdS black holes with $\psi = -5/2$, $N_s = -1$, $\omega_s = -2/3$, and $P = 0.3283$.

Here, r_+ is understood as a function of pressure and temperature ($r_+ = r_+(P, T)$), via the equation of state (15).

3. Perturbations of AdS Black Holes in Rastall Gravity

Now, we study a massless scalar field perturbation on the four-dimensional AdS black holes surrounded by perfect fluid. The test scalar field satisfies the Klein-Gordon equation:

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\Phi(r,t,\theta,\vartheta)\right) = 0. \quad (19)$$

Assuming the scalar field with

$$\Phi(t,r,\theta,\vartheta) = \sum_{lm}\frac{\phi(r)}{r}Y_{lm}(\theta,\vartheta)e^{-i\omega t}, \quad (20)$$

the radial perturbed equation is

$$\left(\frac{d^2}{dr_*^2} + \omega^2 - V(r)\right)\phi(r) = 0, \quad (21)$$

where r_* is the tortoise coordinate, defined by $dr/dr_* = f(r)$. The effective potential $V(r)$ reads as

$$V(r) = f(r)\left(\frac{f'(r)}{r} + \frac{l(l+1)}{r^2}\right), \quad (22)$$

where l is the angular momentum eigenvalue related to the angular momentum operator L^2. We only consider the case of $l = 0$ in this paper.

At the AdS boundary $r \to +\infty$, the generalized potential $V(r)$ diverges, which leads to $\phi(r) \to 0$.

Near the horizon r_+, the scalar field needs to satisfy the ingoing boundary condition, $\phi(r) \sim (r - r_+)^{i\omega/4\pi T}$. Following the same path in Refs. [50, 68–70], we define $\phi(r)$ as $\varphi(r)\exp[-i\int(\omega/f(r))r]$, where $\exp[-i\int(\omega/f(r))dr]$ asymptotically approaches to the ingoing wave near the horizon; then, (21) becomes

$$\varphi''(r) + \varphi'(r)\left(\frac{f'(r)}{f(r)} - \frac{2i\omega}{r} + \frac{2}{r}\right) - \frac{2i\omega}{rf(r)}\varphi(r) = 0. \quad (23)$$

In the limit of $r \to r_+$, we can set $\varphi(r) = 1$ and we have $\varphi(r) = 0$ when $r \to \infty$.

It is worthy to point out that without surrounding perfect fluid ($N_S = 0$), the vacuum AdS black hole solution (11) in the

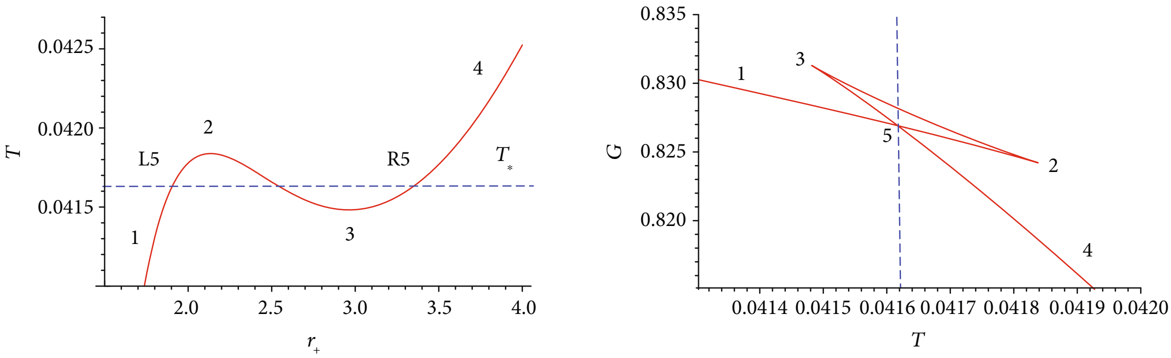


FIGURE 4: The $T - r_+$ and $G - T$ diagrams of four-dimensional AdS black holes with $\psi = -5/2$, $N_s = -1$, $\omega_s = 1/3$, and $P = 0.0328$.

Rastall gravity has the similar form with the Schwarzschild-AdS black hole solution in GR. Under the $l = 0$ scalar field perturbation, the quasinormal modes of Schwarzschild-AdS black hole with the AdS radius $R = 1$ have been computed in Ref. [73], where $n = 0$ fundamental frequency of Schwarzschild-AdS black hole with $r_+ = 0.2$ equals to $2.47511 - 0.38990i$. For the AdS black hole solution (6) in the Rastall gravity, we also set $R_\Lambda = 1$ and evaluate the effect of N_s on the quasinormal frequency, as shown in Figure 2. In the limit of $N_s \to 0$, these QNMs coincide with the fundamental mode $2.47511 - 0.38990i$, which plays the role of a starting point.

By choosing the pressure $P < P_c$, the $T - r_+$ and $G - T$ diagrams of four-dimensional AdS black holes are plotted in Figures 3 and 4. In that case, an inflection point occurs, which displays that the behavior is reminiscent of the Van der Waals system. Notice that the similar diagrams have been presented in Refs. [68, 70], where the point "5" represents coexistence of small and large black hole and the solid lines "1-5" or "1-L5" and "5-4" or "R5-4" separately denote small and large black holes (see Figures 3 and 4). In addition, the points "L5" and "R5" share the same Gibbs free energy and temperature with $T_* \approx 0.13105$ for $\omega_s = -2/3$ and $T_* \approx 0.04162$ for $\omega_s = 1/3$.

On the other hand, the QNM frequencies of massless scalar perturbation against the small and large black holes are shown in Table 2. For the small black hole phase $T < T_*$, the absolute values of the imaginary part of QNM frequencies decrease, while the real part of frequencies increase with the shrink of the horizon radius from T. In the large black hole phase $T < T_*$, the QNM frequencies are characterized by larger real oscillation frequency and larger damping with the increase in the horizon radius. These QNM frequencies for small and large black hole phases are also plotted in Figures 5 and 6. The arrow denotes the increase in the horizon radius r_+.

In addition, the mass M of Schwarzschild AdS black hole from (12) is given as

$$M = \frac{r_+^3}{2R^2} + \frac{r_+}{2}. \tag{24}$$

TABLE 2: The QNM frequencies of (a) ($\omega_s = -2/3$) and (b) ($\omega_s = 1/3$) with the change of temperature T and horizon radius r_+. The italic values denote the small black hole phase, while the rest of the values correspond to the large black hole phase.

(a)

T	r_+	ω
0.1250	*0.275*	*1.66875-0.02303I*
0.1270	*0.280*	*1.66643-0.02462I*
0.1288	*0.285*	*1.66407-0.02629I*
0.1304	*0.290*	*1.66167-0.02804I*
0.1311	1.245	1.97363-0.78752I
0.1312	1.250	1.97610-0.78863I
0.1313	1.255	1.97859-0.78974I
0.1314	1.260	1.98110-0.79083I

(b)

T	r_+	ω
0.040759	*1.700*	*1.15336-0.219889I*
0.041069	*1.750*	*1.14943-0.221412I*
0.041308	*1.800*	*1.14140-0.223724I*
0.041488	*1.850*	*1.13570-0.224786I*
0.041634	3.3525	1.93148-0.263925I
0.041636	3.3550	1.93191-0.263982I
0.041638	3.3575	1.93236-0.264036I
0.041640	3.3600	1.93281-0.264086I

In the limit $r_+ \to 0$, the mass M vanishes and (12) reduces to the pure AdS space:

$$f_{\mathrm{AdS}}(r) = 1 + \frac{r^2}{R^2}, \qquad \frac{1}{R^2} = -\frac{\Lambda}{3}. \tag{25}$$

In other words, the quasinormal modes of small Schwarzschild AdS black holes ($r_+ \to 0$) can tend to the

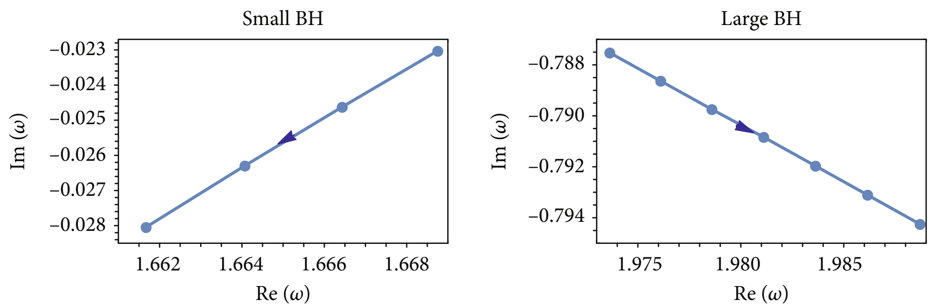


FIGURE 5: The real and imaginary parts of frequencies for large and small black holes with $\omega_s = -2/3$. The arrow denotes the increase in r_+.

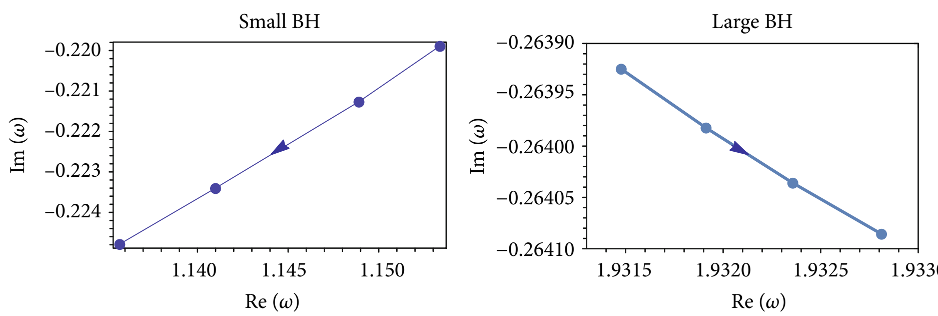
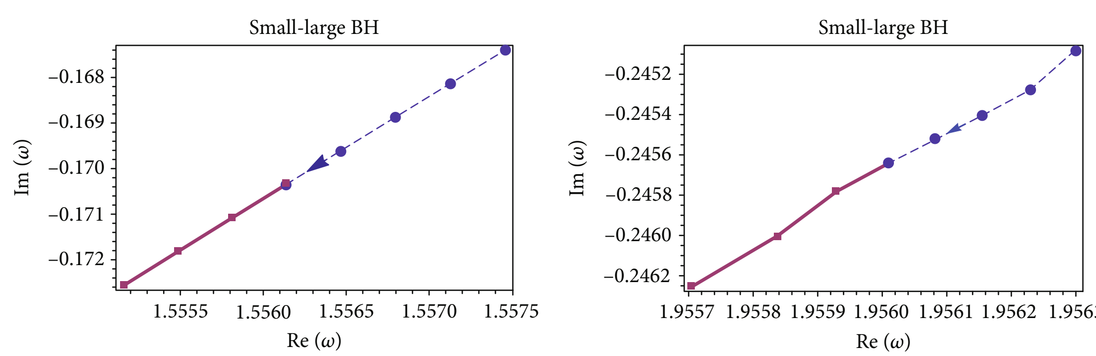


FIGURE 6: The real and imaginary parts of frequencies for large and small black holes with $\omega_s = 1/3$. The arrow denotes the increase in r_+.



(a) $\omega_s = -2/3$

(b) $\omega_s = 1/3$

FIGURE 7: The real and imaginary parts of QNM frequencies for large (dashed) and small (solid) black holes. The arrow denotes the increase in the horizon radius.

purely normal modes of empty AdS spacetime, which has been proven by Konoplya in Ref. [74]. With regard to the AdS black hole (6) in the Rastall theory, the mass is obtained:

$$M_{\mathrm{Ras}} = \frac{r_+^3}{2R_\Lambda^2} + \frac{r_+}{2} - \frac{N_s}{r_+^{\xi-1}}. \tag{26}$$

Evidently, the mass of AdS black hole is divergent in the limit $r_+ \to 0$. Then, this solution (6) cannot reduce the pure AdS space:

$$f_{\mathrm{RAdS}}(r) = 1 + \frac{r^2}{R_\Lambda^2}, \tag{27}$$

because of the existence of $N_s \neq 0$. Therefore, the quasinormal modes of small AdS black holes ($r_+ \to 0$) cannot also tend to the purely AdS spectrum in the Rastall gravity.

In addition, at the critical position $P = P_c$, a second-order phase transition occurs. The QNM frequencies of the small and large black hole phases are shown in Figure 7. The QNM frequencies of two black hole phases display the similar trend of decay as increases in the horizon radius r_+. In fact, this phenomenon has also emerged in some other gravity theories [50, 68–70].

4. Concluding Remarks

In the four-dimensional Rastall theory, we reviewed the $P-V$ criticality and phase transition of AdS black holes in the extended phase space. Considering the weak energy condition of energy density $\rho_s \geq 0$, we derived five proper regions for the parameters ω_s and ψ, where the VdW-like SBH/LBH phase transition could happen for the AdS black holes. Later, we further calculated the QNMs of massless scalar perturbations to probe the SBH/LBH phase transition of AdS black holes surrounded by two special fields: radiation and quintessence fields, respectively. These results reveal that when the SBH/LBH phase transition happens, the slopes of the QNM frequencies change drastically in the small and large black hole phases with the increase in r_+. In other words, the thermodynamic SBH/LBH phase transition has been exactly reflected by the variations of QNM frequencies for corresponding small and large black holes in the four-dimensional Rastall theory.

Nevertheless, this phenomenon does not appear at the critical isobaric phase transitions, where the QNM frequencies for both small and large black holes share the same behavior. This implies that the QNM frequencies are not suitable to probe the black hole phase transition in the second order.

In four-dimensional Rastall gravity, charged Kiselev-like black holes surrounded by perfect fluid have been obtained by Heydarzade and Darabi [9]. It would be interesting to derive charged AdS black hole solutions in the Rastall gravity. Then, we can recover the possible relation between the thermodynamical phase transition and QNM frequencies. Similar discussions for the charged AdS black holes in the Rastall gravity coupled with a nonlinear electric field also deserve a new work in the future.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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